JOURNAL D'ANALYSE MATHÉMATIQUE

RÉDIGÉ ET PUBLIÉ
SOUS LES AUSPICES DU
CONSEIL DE RÉDACTION

PAR

BINYAMIN AMIRÀ

EN COLLABORATION AVEC

ZEEV NEHARI et MENAHEM SCHIFFER

VOLUME IX

JÉRUSALEM
1961/1962

JOURNAL D'ANALYSE MATHÉMATIQUE
(Journal International de Recherches)

Le Journal d'Analyse Mathématique publie des Mémoires originaux d'Analyse Mathématique en français et en anglais.
Les Auteurs reçoivent gratuitement 50 tirés-à-part de leur Mémoire.

Le prix d'abonnement est de 20.— dollars par volume (environ 400 pages). Une remise de 40% est accordée aux Membres *individuels* des Sociétés Mathématiques.

Adresse de la Rédaction:

B. A. Amirà
13, Rue Abrabanel
Jérusalem
Israël

Agence Générale pour Abonnements
General Agents for Subscriptions:
Stechert-Hafner, Inc.
31 East 10th Street
New York 3, N.Y., U.S.A.

Imprimé en Israël
Printed in Israel
Imprimerie "Gutenberg", Z. Casap et fils, S.A., Tel-Aviv

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Les fonds pour la publication de ce volume sont dûs (en partie) à la générosité de

M. Erich Levin
Mme Adina Levin-Vromen
à Tel Aviv

ainsi que

Mr. Irving Jay Fain
à Providence, R. I.
Mr. Alexander Gross
à New York, N. Y.
Mme Rina Spears
à la mémoire du Dr. G. E. Spears

Imprimé en Israël
Printed in Israel
Ipmrimerie "Gutenberg", Z. Casap et fils, S.A., Tel-Aviv

JOURNAL D'ANALYSE MATHÉMATIQUE

The start of this publication was made possible through an initial grant by The Rockefeller Foundation.

TABLE DES MATIÈRES
VOLUME IX

ON NON-LINEAR EQUATIONS IN A COMPLEX BANACH SPACE [1]

By

Leon Brown

in Detroit, Michigan, U.S.A.

In this paper we are concerned with the generalization of known results in the theory of non-linear integral equations and in the theory of several complex variables. Specifically, we are interested in generalizing the Erhardt Schmidt Branching Theory [see 8, 9 and 11] [2] and the Weierstrass Preparation Theorem [see 2, p. 183].

We consider the following problem: let X be a complex Banach space and given a function f, with domain and range in X, which is analytic and bounded for $||x|| \leqq 1$, then what is the nature of the solutions of the functional equation $x - f(x) = y$, where y is a given element of X? This problem has been extensively studied when f is a completely continuous (compact) linear function. An excellent presentation of these results is in Riesz and Sz-Nagy [10].

We wish to analyze the situation when f is a non-linear analytic function with certain conditions. [3] To this end we develop a specific tool, namely, a generalization of the Weierstrass Preparation Theorem.

In paragraph 1 we present some pertinent lemmas in the theory of complex variables.

In paragraph 2 we consider a function f whose domain is in $X \times C$ and range in C, where X is a complex Banach space and C is the space of complex numbers. f is analytic and bounded for $||x|| \leqq 1$, and $|w| \leqq 1$, $x \in X$, $w \in C$. Assuming $f(0, w)$ has an s-fold zero at $w = 0$, then in a neighborhood N of the origin

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1. Most of this paper is pa t of a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the University of Minnesota. The dissertation was written under the direction of Professor Paul C. Rosenbloom whose encouragement and guidance were most helpful.

A portion of this work was done while the writer was at Tulane University where he was partially supported by the National Science Foundation.

2. Numbers in brackets refer to the bibliography at the end of this paper.

3. Jane Cronin considered this problem. Her theory is based on the Leray-Schauder theory of the degree of mapping [see 4, 5].

$$f = \left\{ w^s - \sum_{v=0}^{s-1} H_v(x)\, w^v \right\} \Omega(x, w) = P\Omega .$$

where H_v are analytic functions on X to C, and Ω is a non-zero function on $N \subset X \times C$ to C. The size of this neighborhood is estimated and P and Ω are represented as integrals of the function f.

In paragraph 3, we summarize some results from Riesz's theory of compact linear operators.

In paragraph 4 we consider the functional equation $x - f(x) = y$ where $f(0) = 0$. We assume that f is analytic and $f'(0)$ is a compact transformation. If $(I - f'(0))^{-1}$ exists then the above equation has a unique solution if y is sufficiently small. If $I - f'(0)$ has no inverse then there exist complex valued functions f_i, $i = 1, \ldots, k$ and g on $X \times M$ where M is a finite dimensional subspace of X such that $x = g(y, u) + u$ is a solution if and only if $f_i(y, u) = 0$, $i = 1, \ldots, k$.

Assuming that for each i, $f_i(0, u) \not\equiv 0$, we can apply the Weierstrass Preparation Theorem to these functions and then use the classical elimination theory [see 16] in order to arrive at our "branching equations".

In paragraph 5 we prove that if R is the ring of functions, with domain and range in X, which are analytic at the origin then R is integrally closed. Applying a theorem of Butts, Hall and Mann [3] it is easily seen that a monic polynomial in the polynomial ring of R can be factored uniquely into irreducible monic polynomials.

The sizes of all above mentioned neighborhoods are estimated in this paper.

§1. Lemmas on Analytic Functions

By elementary means one can prove the following well known result:

Lemma 1.1. If $F(z)$ is a non-constant analytic function and $|F(z)| \leqq 1$ for $|z| \leqq 1$ and $|F(0)| = A > 0$, then for $|z| \leqq r < A$,

$$\frac{A - r}{1 - Ar} \leqq |F(z)| \leqq \frac{A + r}{1 + Ar}$$

and

$$|F(z) - F(0)| \leqq \frac{2|z|}{1 + \sqrt{1 - |z|^2}} < 2|z| .$$

We are interested in the nature of the zeros of an analytic function if it is perturbated by a small constant. This is related to the classical result of Hurwitz [15, p. 119].

Lemma 1.2. If $F(z)$ is analytic, and $|F(z)| \leqq 1$ for $|z| \leqq 1$, and F has an s-fold zero at the origin, and $F(z) \neq 0$ for $0 < |z| \leqq 1$, and

$$\eta = \min_{|z|=1} |F(z)| < \max_{|z|=1} |F(z)| \leqq 1,$$

then for $0 < |\lambda| \leqq \eta$, the function $F(z) - \lambda$ has exactly s simple roots $z_i(\lambda)$, $i = 1, \ldots, s$ in $|z| \leqq 1$, and these roots are in the annulus

$$|\lambda|^{1/s} < |z_i| \leq \left(\frac{|\lambda|}{\eta}\right)^{1/s}.$$

The $z_i(\lambda)$ may be so chosen so that they are branches of the same analytic multivalued function with branch cut the negative real axis.

Proof: If $|\lambda| < \eta$, then $|F(z)| \geqq \eta > |\lambda|$ for $|z| = 1$, and by Rouché's theorem [15, p. 146], $F(z) - \lambda$ has exactly s roots in $|z| < 1$.

For

$$|z| < |\lambda|^{1/s} < 1 \ , \ |F(z)| < |z|^s$$

and

$$|F(z) - \lambda| \geqq |\lambda| - |F(z)| > |\lambda| - |z|^s \geqq 0 .$$

Also

$$|F(z)| \geqq \eta\, |z|^s,$$

so that for

$$|z|^s > \left(\frac{|\lambda|}{\eta}\right) \ , \ |F(z) - \lambda| \geqq |F(z)| - |\lambda| \geqq \eta\, |z|^s - |\lambda| > 0 .$$

We proceed to show that the s roots of $F(z) - \lambda$ are distinct, and also construct $z_i(\lambda)$.

Observe that $F(z) z^{-s}$ is analytic and not equal to zero in $|z| \leqq 1$. Let $h(z) = z(F(z) z^{-s})^{1/s}$, where any particular determination of the s-th root is chosen so that $h(z)$ is analytic. For

$|z| \leqq 1$, $|h(z)| = |z|\,|F(z)z^{-s}|^{1/s} = |F(z)|^{1/s} \leqq 1$. $h(0) = 0$, $h(z) \neq 0$ for $0 < |z| \leqq 1$, and for $|z| = 1$, $|h(z)| \geqq \eta^{1/s}$. Therefore, Rouché's theorem implies that for $|t| < \eta^{1/s}$, there is a unique z in $|z| < 1$ such that $h(z) = t$. Let $g(t)$ be the unique z. The implicit function theorem implies that $g(t)$ is analytic for $|t| < \eta^{1/s}$. The equation $h(z) = t$ is equivalent to $F(z) = t^s$.

Placing $t = \lambda^{1/s}$, a branch of the sth root of λ and $w = e^{2\pi i/s}$, we have that $z_j(\lambda) = g(w^{j-1}\lambda^{1/s})$, $j = 1, \ldots, s$ are the s roots of $F(z) - \lambda$. q.e.d.

Note that if $\eta = \max\limits_{|z|=1} |F(z)|$, then $F(z) = \eta z^s$. The zeros of $F(z) - \lambda$ are $\left(\dfrac{\lambda}{\eta}\right)^{1/s}$, and $g(\lambda) = \left(\dfrac{\lambda}{\eta}\right)^{1/s}$ is an analytic multivalued function.

We will have need for the following integral representation:

Lemma 1.3. *If $f(z)$ is analytic for $|z| \leqq 1$ and $f(z) \neq 0$ for $|z| = 1$, and f has exactly s zeros $z_1, \ldots, z_s$ for $|z| < 1$, then for $|z| > 1$,*

$$w(z) = \prod_{n=1}^{s} (z - z_n) = z^s \exp \int_{|\zeta|=1} \frac{f'(\zeta)}{f(\zeta)} \operatorname{Log}\left(1 - \frac{\zeta}{z}\right) d\zeta$$

where $\operatorname{Log} z = \log|z| + i \arg z$, $-\pi < \arg z \leqq \pi$.

Proof: For $|z| > 1$, and $|\zeta| \leqq 1$, $\log\left(1 - \dfrac{\zeta}{z}\right)$ is an analytic function of ζ. One then simply evaluates the above integral with the aid of the theory of residues. q.e.d.

The following lemma is a generalization of the Euclidean Algorithm.

Lemma 1.4. *If $f(z)$, $g(z)$ are analytic for $|z| \leqq 1$ and $f(z) \neq 0$ for $|z| = 1$, and f has exactly s zeros for $|z| < 1$, then there exists a unique polynomial $P(z)$ of degree $< s$, and unique $q(z)$ analytic in $|z| \leqq 1$ such that*

$$g(z) = q(z) f(z) + P(z).$$

Furthermore for $|z| < 1$,

$$P(z) = g(z) - \frac{w(z)}{2\pi i} \int_{|\zeta|=1} \frac{g(\zeta)\, d\zeta}{w(\zeta)(\zeta - z)}$$

and

$$q(z) = \frac{1}{2\pi i} \frac{w(z)}{f(z)} \int\limits_{|\zeta|=1} \frac{g(\zeta)}{w(\zeta)(\zeta - z)} d\zeta .$$

Proof: Case I: We assume f has s simple roots z_j, $j = 1, \ldots, s$. The condition $P(z_j) = g(z_j)$, $j = 1, \ldots, s$ uniquely determines a polynomial of degree $< s$. Let

$$w(z) = \prod_{j=1}^{s} (z - z_j) .$$

Using Lagrange's interpolation formula we find that

$$P(z) = \sum_{j=1}^{s} \frac{g(z_j)\, w(z)}{w'(z_j)(z - z_j)} .$$

If $|z| < 1$ and $z \neq z_i$, $i = 1, \ldots, s$ then $\dfrac{g(\zeta)}{w(\zeta)(\zeta - z)}$ has simple poles at $z_1, \ldots, z_s$ and z. Thus, by the theory of residues

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \frac{g(\zeta)\, d\zeta}{w(\zeta)(\zeta - z)} = - \sum_{j=1}^{s} \frac{g(z_j)}{w'(z_j)(z - z_j)} + \frac{g(z)}{w(z)} .$$

Thus for $z \neq z_i$, $|z| < 1$,

$$P(z) = g(z) - \frac{w(z)}{2\pi i} \int\limits_{|\zeta|=1} \frac{g(\zeta)\, d\zeta}{w(\zeta)(\zeta - z)} .$$

Since P, g and $\dfrac{w(z)}{2\pi i} \displaystyle\int\limits_{|\zeta|=1} \frac{g(\zeta)\, d\zeta}{w(\zeta)(\zeta - z)}$ are analytic functions the above formula is valid for all z such that $|z| < 1$.

Let

$$q(z) = \frac{g(z) - P(z)}{f(z)} = \frac{1}{2\pi i} \frac{w(z)}{f(z)} \int\limits_{|\zeta|=1} \frac{g(\zeta)}{w(\zeta)(\zeta - z)} d\zeta .$$

Since $\dfrac{w(z)}{f(z)}$ is analytic for $|z| \leq 1$, $q(z)$ is analytic for $|z| < 1$. $q(z)$ is uniquely determined since $P(z)$ is uniquely determined. Note that although the above representation of q is only valid for $|z| < 1$, q is analytic for $|z| = 1$.

Case II: Let f have multiple roots. From Lemma 1.2 we see that there exists a δ such that if $0<|\lambda|<\delta$, then $f(z)+\lambda$ has exactly s simple roots $z_j(\lambda)$ for $|z|<1$. Thus for each λ there exists a unique function $q(z,\lambda)$ and polynomial $P(z,\lambda)$ in z of degree less than s, such that

$$g(z) = q(z,\lambda)\,[f(z)+\lambda] + P(z,\lambda).$$

If

$$w(z,\lambda) = \prod_{j=1}^{S} (z - z_j(\lambda))$$

then

$$P(z,\lambda) = g(z) - \frac{w(z,\lambda)}{2\pi i} \int_{|\zeta|=1} \frac{g(\zeta)\,d\zeta}{w(z,\lambda)(\zeta - z)},$$

and

$$q(z,\lambda) = \frac{1}{2\pi i}\,\frac{w(z,\lambda)}{f(z)+\lambda} \int_{|\zeta|=1} \frac{g(\zeta)\,d\zeta}{w(z,\lambda)(\zeta - z)}.$$

One can easily show that

$$\lim_{\lambda\to 0} w(z,\lambda) = w(z,0) = w(z).$$

However we wish to show that $w(z,\lambda)$ is an analytic function of λ. For $|z|>1$, $0<|\lambda|<\delta$

$$w(z,\lambda) = z^s \exp \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f'(\zeta)}{f(\zeta)+\lambda} \operatorname{Log}\left(1 - \frac{\zeta}{z}\right) d\zeta$$
$$= z^s \exp \psi(z,\lambda) \quad \text{(Lemma 1.3)}.$$

If $|z|>1$ and $|\lambda| < \min_{|\zeta|=1} |f(\zeta)| = \eta$ then $\psi(z,\lambda)$ is analytic in λ. Since $w(z,\lambda)$ is a polynomial in z with coefficients functions in λ, each coefficient is analytic in λ and thus for $|\lambda|<\min(\delta,\eta)$, $w(z,\lambda)$ is analytic in λ. Thus for $|\lambda|<\min(\delta,\eta)$, $P(z,\lambda)$ is analytic in λ and therefore

$$\lim_{\lambda\to 0} P(z,\lambda) = P(z,0) = P(z).$$

$P(z)$ is a polynomial of degree less than s since for each λ, $P(z,\lambda)$ is a polynomial of degree less than s.

$$\frac{w(z,\lambda)}{f(z)+\lambda} = \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \frac{w(\zeta,\lambda)\,d\zeta}{(f(\zeta)+\lambda)(\zeta-z)}$$

for $|\lambda| < \eta$ implies that $\dfrac{w(z,\lambda)}{f(z)+\lambda}$ is an analytic function of λ and we have

$$\lim_{\lambda\to 0} q(z,\lambda) = q(z,0) = q(z).$$

We have for

$$|z| < 1\ ,\ g(z) = q(z)\,f(z) + P(z)$$

with the integral representations which are valid. q.e.d.

We will need one more lemma in the theory of analytic functions.

Lemma 1.5. *If for $|z| \leqq 1$, $F(z)$ is analytic and*

$$|F(z)| \leqq 1,\quad F(z) = \sum_{k=s}^{\infty} a_k z^k,\quad a_s \neq 0,\quad s \geqq 1,$$

and for $|z| \leqq 1$, $G(z)$ is analytic and $|G(z)| < \varepsilon$ where

$$0 < \varepsilon \leqq \beta(|a_s|) = \max_{0 \leqq r \leqq |a_s|} \frac{r^s(|a_s| - r)}{1 - |a_s|\, r},$$

then there exists $r_1(\varepsilon) \leqq r_0 \leqq r_2(\varepsilon) \leqq |a_s|$ such that $F+G$ has exactly s zeros in the circle $|z| < r_1(\varepsilon)$ and no zeros in the annulus $r_1(\varepsilon) \leqq |z| \leqq r_2(\varepsilon)$. Furthermore, if $|z| > r_2(\varepsilon)$ and

$$W(z) = z^s \exp \frac{1}{2\pi i} \int\limits_{|\zeta|=r_2(\varepsilon)} \frac{F'(\zeta)+G'(\zeta)}{F(z)+G(z)} \operatorname{Log}\left(1 - \frac{\zeta}{z}\right) d\zeta$$

then W is a polynomial of degree s whose zeros are exactly those of $F+G$ in $|z| \leqq r_2(\varepsilon)$. Thus for

$$|z| \leqq r_2(\varepsilon),\quad \Omega(z) = \frac{F(z)+G(z)}{W(z)}$$

is a non-zero analytic function and $F(z) + G(z) = W(z)\,\Omega(z)$.

We define $r_1(\varepsilon)$, $r_2(\varepsilon)$, $r_0(|a_s|) = r_0$ in the following manner.

1) r_0 is the solution of the equation $r_0^s\left\{\dfrac{|a_s| - r_0}{1 - |a_s|\, r_0}\right\} = \beta(|a_s|)$.

2) $r_1(\varepsilon)$ is equal to the smaller root of $\varepsilon = r^s\left\{\dfrac{|a_s| - r}{1 - |a_s|\, r}\right\}$,
$r_2(\varepsilon)$ is the larger root.

Note that $r_1(\varepsilon)$ is a monotonically increasing function and $r_2(\varepsilon)$ is monotonically decreasing.

Proof: Let $H(z) = \dfrac{F(z)}{z^s}$. $H(z)$ is analytic for $|z| \leqq 1$, and $|z| = 1$, $|H(z)| = |F(z)| \leqq 1$. Thus for $|z| \leqq 1$, $|H(z)| \leqq 1$. $H(0) = a_s \neq 0$. Therefore by Lemma 1.1, we have for $|z| \leqq r \leqq |a_s|$,

$$\frac{|a_s| - r}{1 - |a_s| r} \leqq |H(z)| \leqq \frac{|a_s| + r}{1 + |a_s| r}$$

which is equivalent to

$$|z|^s \left\{ \frac{|a_s| - r}{1 - |a_s| r} \right\} \leqq |F(z)| \leqq |z|^s \left\{ \frac{|a_s| + r}{1 + |a_s| r} \right\}.$$

Let $r = r_1(\varepsilon)$ or $r_2(\varepsilon)$, then for $|z| = r$,

$$|G(z)| < \varepsilon = r^s \left\{ \frac{|a_s| - r}{1 - |a_s| r} \right\} \leqq |F(z)|.$$

Applying Rouché's theorem, we see that $F + G$ has exactly s zeros in $|z| < r_1(\varepsilon)$ and $|z| < r_2(\varepsilon)$, respectively. Consequently $F + G$ has exactly s zeros in $|z| < r_1(\varepsilon)$, and no zeros in $r_1(\varepsilon) \leqq |z| \leqq r_2(\varepsilon)$.

It follows from Lemma 1.3 that

$$W(z) = \prod_{j=1}^{s} (z - z_j)$$

where $\{z_j\}$ are the s roots of $F + G$ in the circle $|z| \leqq r_2(\varepsilon)$. Hence for

$$|z| < r_1(\varepsilon), \quad \Omega(z) = \frac{F(z) + G(z)}{W(z)}$$

is a non-zero analytic function and $F(z) + G(z) = W(z)\,\Omega(z)$.

q.e.d.

§2. The Zeros of Analytic Functions of a Complex Banach Space

Let X, Y and Z denote complex Banach spaces and C the field of complex numbers.

Definition 2.1. Let f be defined in an open set D of X with range in Y. We say f is analytic in D if f is locally

bounded and G-differentiable in D. This is equivalent to f possessing a Fréchet differential at each point in D [see for example 7, pp. 109–112].

Using the concept of Baire continuity Zorn [18] has proven the following generalization of Hartog's theorem:

Theorem 2.2. Let D_1 and D_2 be open sets respectively contained in X and Y. f is a function on $D_1 \times D_2 \subset X \times Y$ to Z. Note that $X \times Y$ is a complex Banach space with $\|(x, y)\| = \|x\| + \|y\|$. If for each $x \in D_1$, f is analytic in D_2 and for each $y \in D_2$, f is analytic in D_1, then f is analytic in $D_1 \times D_2$.

We further generalize the Euclidean Algorithm.

Theorem 2.3. Let f and g be two functions, with domain in $X \times C$ and range in C, analytic for $\|x\| \leqq 1$, $|w| \leqq 1$. If for each x such that $\|x\| \leqq 1$, $f(x, w)$ has exactly s zeros, $w_j(x)$, $j = 1, \ldots, s$ in $|w| < 1$, and $f(x, w) \neq 0$ for $|w| = 1$, then there exists uniquely a polynomial P in w of degree $< s$ with functions on X to C as coefficients which are analytic for $\|x\| < 1$, and a function Q on $X \times C$ to C which is analytic for $\|x\| < 1$, $|w| < 1$ such that for $\|x\| < 1$, $|w| < 1$,

$$g(x, w) = Q(x, w) f(x, w) + P(x, w).$$

In addition we have for $|w| > 1$,

$$W(x, w) = \prod_{j=1}^{s} (w - w_j(x)) = w^s \exp \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f_w(x, \zeta)}{f(x, \zeta)} \operatorname{Log}\left(1 - \frac{\zeta}{w}\right) d\zeta$$

and for $\|x\| < 1$, $|w| < 1$,

$$Q(x, w) = \frac{1}{2\pi i} \frac{W(x, w)}{f(x, w)} \int_{|\zeta|=1} \frac{g(x, \zeta)\, d\zeta}{W(x, \zeta)(\zeta - w)}$$

and

$$P(x, w) = g(x, w) - \frac{W(x, w)}{2\pi i} \int_{|\zeta|=1} \frac{g(x, w)\, d\zeta}{W(x, \zeta)(\zeta - w)}.$$

Proof: By Lemma 1.3 if $|w| > 1$ for each x,

$$W(x, w) = w^s \exp \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f_w(x, \zeta)}{f(x, \zeta)} \operatorname{Log}\left(1 - \frac{\zeta}{w}\right) d\zeta .$$

Since $W(x, w)$ is a polynomial in w, in order to prove W is analytic, it is sufficient to prove that for each w such that $|w| > 1$ $W(x, w)$ is analytic in x. For each w, with $|w| > 1$, and for every $h \in X$ with λ sufficiently small,

$$W(x + \lambda h, w) = w^s \exp \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f_w(x+\lambda h, \zeta)}{f(x+\lambda h, \zeta)} \operatorname{Log}\left(1 - \frac{\zeta}{w}\right) d\zeta$$

and thus is an analytic function of λ, for $\lambda = 0$. Consequently $W(x, w)$ is G-differentiable with respect to x. Since $f(x, \zeta) \neq 0$ for $|\zeta| = 1$, and $\{\zeta \in C \mid |\zeta| = 1\}$ is a compact set, $W(x, w)$ is a continuous function in x, and thus locally bounded. Therefore $W(x, w)$ is an analytic function of x, which implies that W is an analytic function of (x, w) for $\|x\| < 1$, $w \in C$.

From Lemma 1.4, we see that there exist uniquely two functions, Q and P, namely

$$Q(x, w) = \frac{1}{2\pi i} \frac{W(x, w)}{f(x, w)} \int_{|\zeta|=1} \frac{g(x, \zeta)\, d\zeta}{W(x, \zeta)(\zeta - w)}$$

and

$$P(x, w) = g(x, w) - \frac{W(x, w)}{2\pi i} \int_{|\zeta|=1} \frac{g(x, \zeta)\, d\zeta}{W(x, \zeta)(\zeta - w)},$$

such that $g = Qf + P$. To complete the proof we need to prove that P and Q are analytic for $\|x\| < 1$ and $|w| < 1$.

For $\|x\| < 1$, $|w| < 1$, g and W are analytic. A proof similar to the one for W shows that $\displaystyle\int_{|\zeta|=1} \frac{g(x, \zeta)\, d\zeta}{W(x, \zeta)(\zeta - w)}$ is analytic in each variable and thus analytic (Theorem 2.2). Consequently P is analytic.

In order to prove that Q is analytic it is sufficient to prove that $\dfrac{W}{f}$ is analytic. For each x, $\|x\| < 1$, $\dfrac{W(x, w)}{f(x, w)}$ is an analytic function of w. Thus

$$\frac{w(x, w)}{f(x, w)} = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{W(x, \zeta)\, d\zeta}{f(x, \zeta)(\zeta - w)} .$$

Therefore W/f is analytic for $\|x\| < 1$ and $|w| < 1$. q.e.d.

The main theorem of this section is the generalization of the Weierstrass Preparation Theorem.

Theorem 2.4. Let f be a function on $X \times C$ to C analytic and bounded in $\overline{D} = \{(x, w) \mid \|x\| \leqq 1, \ |w| \leqq 1\}$, $|f(x, w)| \leqq 1$ there and

$$f(0, w) = \sum_{k=s}^{\infty} a_k w^k, \quad a_s \neq 0, \quad s \geqq 1 ;$$

then there exist uniquely $s+1$ functions $H_j(x)$, $j = 0, \ldots, s-1$, $\Omega(x, w)$ analytic in $|w| < r < |a_s|$,

$$\|x\| \leqq g(r) = \frac{r^s}{2}\left(\frac{|a_s| - r}{1 - |a_s| r}\right),$$

and $H_j(0) = 0$, $\Omega(0, 0) = a_s$, $\Omega(x, w) \neq 0$ in this neighborhood, and

$$f(x, w) = \left(w^s - \sum_{j=0}^{s-1} H_j(x) w^j\right) \Omega(x, w) = P\Omega .$$

Furthermore if r_0 is the solution of

$$g(r_0) = \alpha(|a_s|) = \max_{0 \leqq r \leqq |a_s|} g(r),$$

and if for $0 = \rho \leqq \alpha(|a_s|)$, $G_1(\rho)$ is equal to the smaller root of $g(r) = \rho$, and $G_2(\rho)$ is equal to the larger root (for $\rho = \alpha(|a_s|)$, $G_1 = G_2 = r_0$), then for $\|x\| \leqq \rho \leqq \alpha(|a_s|)$, $f(x, w)$ has exactly s zeros in $|w| \leqq G_1(\rho)$ and none in $G_1(\rho) \leqq |w| \leqq G_2(\rho)$. Thus for $\|x\| \leqq \rho \leqq \alpha(|a_s|)$, and $|w| \leqq G_2(\rho)$, we have $f = P\Omega$ and P and Ω are analytic.

We have for $\|x\| \leqq G(r)$ and $|w| \leqq r \leqq |a_s|$,

$$|(Px, w)| \leqq r^s \left(1 + \frac{1}{2^{1/s}}\right)^s$$

and

$$|\Omega(x, w)| > \frac{1}{(2^{1/s} + 1)^s} \left(\frac{|a_s| - r}{1 - |a_s| r}\right) > 0 .$$

Observe that

$$\max_{0\leqq r\leqq |a_s|} \frac{r^s(|a_s|-r)}{2} \leqq \alpha(|a_s|) \leqq \max_{0\leqq r\leqq |a_s|} \frac{r^s}{2}\left(\frac{|a_s|-r}{1-|a_s|^2}\right);$$

therefore

$$\frac{s^s}{2(s+1)^{s+1}}|a_s|^{s+1} \leqq \alpha(|a_s|) \leqq \frac{s^s}{2(s+1)^{s+1}} \frac{(|a_s|)^{s+1}}{(1-|a_s|^2)}.$$

Also if $r \leqq |a_s| < 1$, then

$$\frac{|a_s|-r}{1-|a_s|\,r} \leqq |a_s|.$$

Consequently,

$$\alpha(|a_s|) \leqq \frac{|a_s|}{2} \max_{0\leqq r\leqq |a_s|} r^s = \frac{|a_s|^{s+1}}{2} < \frac{1}{2}.$$

Proof: Let $g(\lambda) = f\left(\lambda \dfrac{x}{\|x\|}, w\right)$. We see that for $|\lambda| \leqq 1$, $g(\lambda)$ is analytic and $|g(\lambda)| \leqq 1$. Applying Lemma 1.1, we have for $|\lambda| \leqq 1$,

$$|g(\lambda) - g(0)| \leqq \frac{2}{1+\sqrt{1-|\lambda|^2}} < 2|\lambda|.$$

Setting $\lambda = \|x\|$, we have $|g(\|x\|) - g(0)| = |f(x, w) - f(0, w)| < 2\|x\|$. We let $F(w) = f(0, w)$ and $G(w) = f(x, w) - f(0, w)$ and $\varepsilon = 2\|x\|$ and apply Lemma 1.5. We have, if $\|x\| \leqq \rho \leqq \alpha(|a_s|) = \dfrac{\beta(|a_s|)}{2}$, $f(x, w)$ has exactly s zeros in $|w| < G_1(\rho)$, and none in

$$G_1(\rho) \leqq |w| \leqq G_2(\rho).$$

Furthermore if for $|w| > G_2(\rho)$, $\|x\| \leqq \rho$,

$$W(x, w) = w^s \exp \frac{1}{2\pi i} \int_{|\zeta|=G_2(\rho)} \frac{f_w(x, \zeta)}{f(x, \zeta)} \operatorname{Log}\left(1-\frac{\zeta}{w}\right) d\zeta$$

then W is a polynomial in w whose zeros are exactly those of $f(x, w)$. With the aid of this representation of $W(x, w)$ we have that W is analytic for $\|x\| < \rho$, $|w| < G_2(\rho)$. Let $\Omega(x, w) = \dfrac{f(x, w)}{W(x, w)}$. We have for $\|x\| < \rho$, $|w| < G_2(\rho)$,

$$\Omega(x, w) = \frac{1}{2\pi i} \int_{|\zeta|=G_2(\rho)} \frac{f(x, \zeta)}{W(x, \zeta)} \frac{1}{\zeta - w} d\zeta .$$

Note that for $|\zeta| = G_2(\rho)$, $W(x, \zeta) \neq 0$. Consequently $\Omega(x, w)$ is analytic there. We define $H(x)$ by the equation

$$P(x, w) = W(x, w) = w^s - \sum_{j=0}^{s-1} H_j(x) w^j .$$

Since $W(0, w) = w^s$, $H_j(0) = 0$, $j = 0, 1, \ldots, s-1$. The analyticity of W implies that H_j is analytic for $\|x\| < \rho < \alpha(|a_s|)$, $|w| < G_2(\rho)$.

In order to complete our estimate we need the following result: for $\|x\| \leqq g(r)$, $\frac{r}{2^{1/s}} < |w| \leqq r$, $f(x, w) \neq 0$.

Let $h(A) = \frac{A-r}{1-Ar}$, $0 < r < |a_s| < 1$. For $A < 1$, $h(A)$ is strictly monotonically increasing because $h'(A) = \frac{1-r^2}{(1-Ar)^2} > 0$ for $r > 1$.

We know that for $0 < |w| \leqq r$, $|f(0, w)| \leqq |w|^s \frac{|a_s| - r}{1 - |a_s| r} > 0$.

Applying Lemma 1.1 to $f\left(\lambda \frac{x}{\|x\|}, w\right)$ and then setting $\lambda = \|x\|$, we have for $w \neq 0$, and $\|x\| \leqq r_1 < |f(0, w)|$,

$$|f(x, w)| > \frac{|f(0, w)| - r_1}{1 - |f(0, w)| r_1}$$

$$\geqq \frac{|w|^s \left\{\frac{|a_s| - r}{1 - |a_s| r}\right\} - r_1}{1 - |w|^s \left\{\frac{|a_s| - r}{1 - |a_s| r}\right\} r_1} .$$

If

$$|w| > \frac{r}{2^{1/s}} \quad \text{and} \quad r_1 = \frac{r^s}{2} \left(\frac{|a_s| - r}{1 - |a_s| r}\right) = g(r) < |f(0, w)| ,$$

we have

$$|w|^s \left(\frac{|a_s| - r}{1 - |a_s| r}\right) - r_1 > \frac{r^s}{2} \left(\frac{|a_s| - r}{1 - |a_s| r}\right) - r_1 = 0 .$$

So we have, if $\|x\| \leqq g(r)$ and $\dfrac{r}{2^{1/s}} \leqq |w| \leqq r$, $f(x, w) \neq 0$.

Consequently, for $\|x\| \leqq g(r)$, $|w| \leqq r$,

$$|P(x, w)| = |W(x, w)| = \left|\prod_{j=1}^{s} (w - w_j(x))\right| \leqq r^s \left(1 + \frac{1}{2^{1/s}}\right)^s .$$

$$|\Omega(x, w)| = \left|\frac{f(x, w)}{W(x, w)}\right| .$$

Applying the minimum modulus theorem we have

$$\begin{aligned} |\Omega(x, w)| &\geqq \min_{|w|=r} |\Omega(x, w)| \\ &\geqq \frac{\min_{|w|=r} |f(x, w)|}{\max_{|w|=r} |W(x, w)|} \geqq \frac{\min_{|w|=r} |f(x, w)|}{r^s\left(1 + \frac{1}{2^{1/s}}\right)^s} . \end{aligned}$$

We have shown earlier that for $|w| = r$, and for $\|x\| \leqq r_1 < |f(0, w)|$,

$$|f(x, w)| \geqq \frac{r^s\left(\dfrac{|a_s| - r_1}{1 - |a_s| r}\right) - r_1}{1 - r^s\left(\dfrac{|a_s| - r}{1 - |a_s| r}\right) r_1} = \frac{2g(r) - r_1}{1 - 2g(r) r_1},$$

Let $r_1 = g(r)$, then for $|w| = r$, $\|x\| \leqq g(r)$,

$$|f(x, w)| \geqq \frac{g(r)}{1 - 2(g(r))^2} > g(r)$$

since

$$g(r) \leqq \alpha(|a_s|) < \frac{1}{2} .$$

Therefore

$$|\Omega(x, w)| > \frac{g(r)}{r^s\left(1 + \dfrac{1}{2^{1/s}}\right)^s} = \frac{|a_s| - r}{1 - |a_s| r} \cdot \frac{1}{(2^{1/s} + 1)^s} .$$

q.e.d.

Observe that the qualitative part of Theorem 2.4 can be proved by applying Theorem 2.3. This would give us a different integral representation of P.

We introduce the following notation: if A is any open sphere in C, then $H_\infty(A) = \{f \mid f$ analytic and bounded in A with $\|f\| = \sup_{z \in A} |f(z)|\}$. If $A = \{z \in C \mid |z| < 1\}$, we write H_∞ instead of $H_\infty(A)$ and let

$$S = \{f \in H_\infty \mid \|f\| \leq 1\}.$$

Theorem 2.5. Let T be a topological space and g a continuous function from T into S. If

$$f(t_0)(z) = \sum_{k=s}^{\infty} a_k z^k, \quad a_s \neq 0, \quad s \geq 1,$$

then there exists a neighborhood of t_0, N, an open sphere in C, A_0, with center at 0, s continuous functions $H_j(t)$, $j=0, \ldots, s-1$ from N to C and a continuous function Ω from N to $H_\infty(A_0)$ such that for all $t \in W$, $z \in A_0$, $\Omega(t)(z) \neq 0$ and

$$f(t)(z) = \left(z^s - \sum_{j=0}^{s-1} H_j(t) z^j\right) \Omega(t)(z).$$

Proof: Using Lemma 1.5 the proof is essentially the same as the first part of the proof of Theorem 2.4.

§ 3. Summary of Known Results on Banach Spaces

We summarize some results in Riesz's theory of compact operators (See Riesz and Sz-Nagy [10]).

Let X be a Banach space and $T = I - K$ where K is a linear compact transformation on X to X. If $A \subset X$ then

$$T^{-1}(A) = \{x \in X \mid T(x) \in A\}$$

and

$$T(A) = \{y \in X \mid \exists\, x \in A \ni T(x) = y\}.$$

Definition 3.1. $M_0 = \{0\}$ and $M_n = T^{-1}(M_{n-1})$. $N_0 = X$ and $N_n = T(N_{n-1})$.

Theorem 3.2. M_n are finite dimensional subspaces of X, and N_n are closed and thus subspaces of X.

Theorem 3.3. There exists an integer v such that

$$M_0 \overset{\subset}{\neq} M_1 \overset{\subset}{\neq} M_2 \overset{\subset}{\neq} \ldots \overset{\subset}{\neq} M_v = M_{v+1} = \ldots,$$

and

$$N_0 \overset{\supset}{\neq} N_1 \overset{\supset}{\neq} \ldots \overset{\supset}{\neq} N_v = N_{v+1} + \ldots,$$

and T restricted to N_v is an invertible operator.

Theorem 3.4. $X = M_v \oplus N_v$; that is, if $x \in X$ there exists uniquely $u \in M_v$ and $w \in N_v$ such that $x = u + w$.

Theorem 3.5. There exists $x_{ij} \in X$, a basis for M_v, such that

$$T(x_{ij}) = \begin{cases} x_{i,j+1} & j = 1, \ldots, v_i - 1 \\ 0 & j = v_i \end{cases}$$

where $v = v_1 \geqq v_2 \geqq \ldots \geqq v_k$, and $\sum_{j=1}^{k} v_j = n =$ dimension of M_v.

We now consider the adjoint transformation $T^* = (I-K)^* = I - K^*$. Since K is a compact transformation K^* is a compact transformation on X^* to X^* and we define M_n^* and N_n^* as we defined M_n and N_n. One easily proves that:

Lemma 3.6. $M_n^* = (M_n)^*$.

Lemma 3.7. There exists x_{ij}^* a basis of M_v^* such that $x_{ij}^*(x_{kl}) = \delta_{ik}\delta_{jl}$ where x_{kl} are the basis elements of M_v.

A simple computation and we have

Lemma 3.8.

$$T^*(x_{ij}^*) = \begin{cases} x_{i,j-1}^* & j = 2, \ldots, v_i \\ 0 & j = 1. \end{cases}$$

§4. Non-Linear Equations and the Schmidt Branching Equations

Let X be a complex Banach space and f a function on X to X which is analytic at the origin. Let $f(0) = 0$ and $f'(0)$, which is a linear transformation from X to X, be a compact operator. We then consider the functional equation $(I - f)(x) = y$ where x and y are in X. If $F(x) = f(x) - f'(0)x$, then $F(0) = 0$ and $F'(0)(x) \equiv 0$. In this situation $(I - f)(x) = y$ is equivalent to $(I - f'(0))(x) = y + F(x)$. If $(I - f'(0))^{-1}$ exists, we have $x = (I - f'(0))^{-1}(y + F(x))$ which has a unique solution for all y in a neighborhood of the origin (see fixed point theorem;

Theorem 4.3 or [6]). The more interesting situation occurs if the inverse of $I - f'(0)$ does not exist.

Consider first the linear case: $(I - f'(0))(x) = y$, and let $T = I - f'(0)$ We use the notation introduced in paragraph 3. We have

$$M_1 = T^{-1}(0) = \{x \in X \mid x^*_{j1}(x) = 0,\ j = 1, \ldots, k\}$$

since

$$(T^*)^{-1}(0) = \{x^*_{j1},\ j = 1, \ldots, k\}$$

and the range of T is closed (see [1]). Let

$$\mathfrak{X} = \{x \in X \mid X^*_{iv_i}(x) = 0\,,\ i = 1, \ldots, k\}\,.$$

Theorem 4.1. *There exists a* $W \in X^X$ *such that*

$$TW = I - \sum_{i=1}^{k} x_{i1} \otimes x^*_{i1} \quad \text{and} \quad W(X) \subset \mathfrak{X}\,.$$

Consequently $TW(x) = x$ *for all* $x \in N_1$.

Proof: Let $P = \sum_{i,j} x_{ij} \otimes x^*_{ij}$ and $P' = I - P$. We have $P^2 = P$, $(P')^2 = P'$, and for each $x \in X$, $x = P(x) + P'(x)$ where $P(x) \in M_v$ and $P'(x) \in N_v\,(X = M_v \oplus N_v)$.

$T(N_v) = N_v$ and $T^{-1} \mid N_v$ exists. Let $V = T^{-1} \mid N_v : V \in N_v^{N_v}$.

Define $W = Q + VP'$ where

$$Q = \sum_{i=1}^{k} \sum_{j=1}^{v_i - 1} (x_{ij} \otimes x^*_{i,j+1})\,.$$

For $x \in X$, we have

$$x = P(x) + P'(x) = \sum_{i=1}^{k} \sum_{j=1}^{v_i} b_{ij}\, x_{ij} + P'(x)$$

and

$$\begin{aligned} W(x) &= (Q + VP')(x) = (Q + VP')(P(x) + P'(x)) \\ &= QP(x) + QP'(x) + VP'P(x) + VP'P(x) \\ &= QP(x) + VP'(x) \\ &= \left\{\sum_{i=1}^{k} \sum_{j=1}^{v_i - 1} (x_{ij} \otimes x^*_{i,j+1})\right\} \left(\sum_{i=1}^{k} \sum_{j=1}^{v_i} b_{ij}\, x_{ij}\right) + VP'(x) \end{aligned}$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{v_i - 1} b_{i,j+1} x_{ij} + VP'(x) \in \mathfrak{X}.$$

$$TW(x) = \sum_{i=1}^{k} \sum_{j=1}^{v_i - 1} b_{i,j+1} x_{i,j+1} + P'(x)$$

$$= \sum_{i=1}^{k} \sum_{j=2}^{v_i} b_{ij} x_{ij} + P'(x)$$

$$= P(x) - \sum_{i=1}^{k} b_{i1} x_{i1} + P'(x)$$

$$= x - \left\{ \sum_{i=1}^{k} (x_{i1} \otimes x_{i1}^*) \right\} (x).$$

q.e d.

Corollary 4.2: The general solution of $T(x) = y$, where $y \in N_1$ is $x = W(y) + u$ where $u \in M_1$.

Proof: If y is not in $N_1 = T(X)$ then there does not exist any solutions of $T(x) = y$. The condition $y \in N_1$ is equivalent to the conditions $x_{i1}^*(y) = 0$ for $i = 1, \ldots, k$.

We return to our original problem: $x - f(x) = y$ which is equivalent to $x - f'(0)x = y + F(x)$. We have: there exist solutions if and only if $x_{i1}^*(y + F(x)) = 0$ for $i = 1, \ldots, k$ and these solutions are

$$x = W(y + F(x)) + u$$

where $u \in M_1$. If we let $x = x_1 + u$ where $x_1 \in \mathfrak{X}$ and $u \in M_1$ we have

1) $x_1 = W(y + F(x_1 + u))$, and
2) $x_{i1}^*(y + F(x_1 + u)) = 0 \quad i = 1, \ldots, k$.

From equation 1 it is seen that a fixed point theorem is indicated.

Theorem 4.3: A Fixed Point Theorem (Hildebrandt and Graves [6]). Let X be a complete metric space and

$$S(x_0, a) = \{x \in X \mid \rho(x, x_0) \leqq a\}.$$

Let f be a function on $S(x_0, a)$ to X such that

1) A Lipshitz condition is satisfied; that is $\rho(f(x_1), f(x_2)) \leqq k\rho(x_1, x_2)$ and $0 < k < 1$.

2) $\rho(x_0, f(x_0)) = b \leqq a(1-k)$,

then there exists a unique $x \in S\left(x_0, \frac{b}{1-k}\right) \subset S(x_0, a)$ such that $f(x) = x$ ($f(x) \neq x$ for all other $x \in S(x_0, a)$).

Lemma 4.4: Let $y \in S(x_0, a)$ such that $\rho(f(y), y) \leqq \varepsilon$, then, if x is the solution of $f(x) = x$ in Theorem 4.3,

$$\rho(x, y) = \frac{\varepsilon}{1-k}.$$

Proof:

$\rho(x, y) \leqq \rho(x, f(x)) + \rho(f(x), f(y)) + \rho(f(y), y)) \leqq 0 + k\rho(x, y) + \varepsilon$.

Therefore

$$\rho(x, y) = \frac{\varepsilon}{1-k}.$$

q.e.d.

For Lemmas 4.5 to 4.9 we assume that F is a function from one complex Banach space X to another Y.

Lemma 4.5: If F is analytic for $\|x\| \leqq r$ and $F(0) = 0$, $\|F(x)\| \leqq M$ for $\|x\| \leqq r$, then for

$$\|x\| \leqq r, \quad \|F(x)\| \leqq M \frac{\|x\|}{r}.$$

Proof: For a fixed $x \in X$, there exists a $y^* \in Y^*$ such that $\|y^*\| = 1$ and $y^*(F(x)) = \|F(x)\|$ (Hahn-Banach Theorem [1]). Let

$$G(\lambda) = y^*\left(F\left(\lambda \frac{x}{\|x\|}\right)\right).$$

$G(0) = y^*(F(0)) = y^*(0) = 0$ and for $\|\lambda\| \leqq r$,

$$|G(\lambda)| \leqq \|y^*\| \left\|F\left(\lambda \frac{x}{\|x\|}\right)\right\| \leqq M.$$

Therefore

$$|G(\lambda)| \leqq M \frac{|\lambda|}{r}$$

for $|\lambda| \leqq r$ (Schwartz's lemma). Setting $\lambda = \|x\|$, we have the desired result.

Lemma 4.6: If F is analytic for $\|x\| \leqq 1$ and $F(0) = 0$, $\|F(x)\| \leqq M$ for $\|x\| \leqq 1$, then for

$$\|x\| < 1, \quad \|F'(x)\| \leqq \frac{M}{1 - \|x\|}.$$

Proof: Let $x_0 \in X$ and $x \in X$ such that $\|x\| = 1$, then there exists a $y^* \in Y^*$ such that $\|y^*\| = 1$ and $y^*(F'(x_0)x) = \|F'(x_0)x\|$. Let $G(\lambda) = y^*(F(x_0 + \lambda x))$ for $|\lambda| \leqq 1 - \|x_0\|$.

$$|G(\lambda)| \leqq \|y^*\| \; \|F(x_0 + \lambda x)\| \leqq M$$

which implies that

$$|G'(0)| \leqq \frac{M}{1 - \|x_0\|}$$

(Cauchy's Inequality). Thus

$$G'(0) = y^*(F'(x_0)x) = \|F'(x_0)x\| \leqq \frac{M}{1 - \|x_0\|}.$$

Hence

$$\|F'(x_0)\| = \sup_{\|x\|=1} \|F'(x_0)x\| \leqq \frac{M}{1 - \|x_0\|}.$$

q.e.d.

Lemma 4.7: If in addition to the assumptions of Lemma 4.6 we assume $F'(0)(x) \equiv 0$, then

$$\|F'(x)\| \leqq M \frac{\|x\|}{r} \cdot \frac{1}{1-r}$$

for $\|x\| \leqq r < 1$.

Proof: For

$$\|x\| \leqq r, \quad \|F'(x)\| \leqq \frac{M}{1 - \|x\|} \leqq \frac{M}{1-r}$$

(Lemma 4.6). Since $F'(x)$ is an analytic function from X to Y^X, Lemma 4.5 implies that

$$\|F'(x)\| \leqq \frac{M}{1-r} \; \frac{\|x\|}{r}.$$

q.e.d.

Lemma 4.8: With the same assumptions as in Lemma 4.7, we find that

$$\|F'(x)\| \leqq \begin{cases} 4\,\|x\|\,M & \text{for} \quad \|x\| \leqq 1/2 \\ \dfrac{M}{1-\|x\|} & \text{for} \quad \|x\| \geqq 1/2 . \end{cases}$$

Proof: Note that $\dfrac{1}{(1-r)\,r}$ has a minimum at $r=\frac{1}{2}$. We obtain the desired results by Lemma 4.8, setting $r=\frac{1}{2}$ when $\|x\| \leqq \frac{1}{2}$ and $r=\|x\|$ when $\|x\| \geqq \frac{1}{2}$.

Lemma 4.9: With the same assumptions as in Lemma 4.7, we find that if $\|x_1\|$, $\|x_2\| \leqq r \leqq \frac{1}{2}$, then

$$\|F(x_1) - F(x_2)\| \leqq 4rM\,\|x_1 - x_2\| .$$

Proof: If $\|x\| \leqq r \leqq \frac{1}{2}$, then $\|F'(x)\| \leqq 4M\,\|x\| \leqq 4rM$ (Lemma 4.8). Let x_1 and x_2 be given, $\|x_1\|$, $\|x_2\| \leqq r$. There exists a $y^* \in Y^*$ such that

$$y^*(F(x_1) - F(x_2)) = \|F(x_1) - F(x_2)\| .$$

Let $G(\lambda) = y^*(F(x_1 + \lambda(x_2 - x_1)))$ for $|\lambda| \leqq 1$.

$$G(1) - G(0) = y^*(F(x_2) - F(x_1)) = \|F(x_2) - F(x_1)\| .$$

$$\|x_1 - \lambda(x_2 - x_1)\| = \|(1-\lambda)\,x_1 + \lambda x_2\| \leqq |1-\lambda|\;\|x_1\| + |\lambda|\;\|x_2\|$$
$$\leqq |1-\lambda|\,r + |\lambda|\,r = r \quad \text{for} \quad 0 \leqq \lambda \leqq 1 .$$

We have

$$G(1) - G(0) = \int_0^1 G'(\lambda)\,d\lambda \leqq \operatorname*{Max}_{0\leqq\lambda\leqq1} |G'(\lambda)|$$
$$= \operatorname*{Max}_{0\leqq\lambda\leqq1} |y^*[F'(x_1 + \lambda(x_2 - x_1))\,(x_2 - x_1)]|$$
$$\leqq \operatorname*{Max}_{0\leqq\lambda\leqq1} \|y^*\|\;\|F'(x_1 + \lambda(x_2 - x_1))\|\;\|x_2 - x_1\|$$
$$\leqq 4rM\,\|x_2 - x_1\| .$$

q.e.d.

Lemma 4.10: If $K \in X^X$ is a compact transformation then $\|I - K\| \geqq 1$. Furthermore, if $(I-K)^{-1}$ exists, then $\|(I-K)^{-1}\| \geqq 1$.

Proof: There exist x_n such that $\|x_n\| = 1$ and $\|K(x_n)\| \leqq 1/n$. If not, let $a = \min\limits_{\|x\|=1} \|K(x)\| > 0$. Thus for $x \in K(X)$, K^{-1} exists, and $\|K^{-1}(x)\| \leqq \frac{1}{a}\|x\|$. For any bounded set $B \subset K(X)$, $K^{-1}(B)$ is bounded and hence $\overline{K(K^{-1}(B))}$ is compact. $K(X)$ is locally compact and therefore is of finite dimension. This implies K is of finite rank which is a contradiction.

We have

$$\|I-K\| = \sup_{\|x\|=1} \|x - K(x)\| \geqq \|x_n - K(x_n)\|$$
$$\geqq \|x_n\| - \|K(x_n)\| \geqq 1 - \frac{1}{n}.$$

Therefore $\|I-K\| \geqq 1$.

If $(I-K)^{-1}$ exists, then let $(I-K)^{-1} = I - \overline{K}$. We have $(I-K)(I-\overline{K}) = I$ or $\overline{K} = (I-K)^{-1}(-K)$. Consequently $\overline{K}$ is a compact transformation and $\|(I-K)^{-1}\| = \|I-\overline{K}\| \geqq 1$.

q.e.d

Lemma 4.11: If

$$W = Q + VP' = \sum_{i=1}^{k} \sum_{j=1}^{v_i-1} (x_{ij} \otimes x^*_{i,j+1}) + VP'$$

(see Theorem 4.1). Then $\|W\| \geqq 1$.

Proof: $\|W\| \geqq \|W\|_{N_v} = \|VP'\|_{N_v} = \|V\|_{N_v} \geqq 1$ (Lemma 4.10 and $P' = I$ on N_v).

q.e.d.

Let us return to our original problem. We have

$$x_1 = W(y + F(x_1 + u)),$$

where $u \in M_1$, $F(0) = 0$ and $F'(0) = 0$. We make the additional assumption that for $\|x\| \leqq 1$, $\|f(x)\| \leqq 1$. This implies that for $\|x\| \leqq 1$,

$$\|F(x)\| = \|f(x) - f'(0)x\| \leqq \|f(x)\| + \|f'(0)\| \, \|x\| \leqq 1 + \|x\| \leqq 2.$$

Therefore, $\|F(x)\| \leqq 2(\|x\|)^2$. We now apply the fixed point theorem.

Theorem 4.12: For fixed y and u, let $G(x) = W(y + F(x+u))$. Let $g(r) = r(1 - 8r\|W\| - 8\rho\|W\|)$, where $\rho < \dfrac{1}{16\|W\| + 32\|W\|^2}$

and let r_0 be the solution of $g(r_0) = \max_{0 \leqq r \leqq 1} g(r)$. Then for $\|y\|$, $\|u\| < \rho$, there exists a unique x_1 in the sphere $\|x_1\| < r_1$ such that $G(x_1) = x_1$, and no fixed point in the annullus $r_1 \leqq \|x\| \leqq r_0$, where r_1 is the solution of $g(r) = (\rho + 2\rho^2)\|W\|$ and $0 < r_1 < r_0$.

Proof: By the use of elementary calculus we find r_0 and $g(r_0)$. We have

$$r_0 = \frac{1 -- 8\|W\|}{16\|W\|} \quad \text{and} \quad g(r_0) = \frac{(1 - 8\rho\|W\|)^2}{32\|W\|} .$$

Thus

$$r_0 + \rho = \frac{1 + 8\rho\|W\|}{16\|W\|} < \frac{3 + 4\|W\|}{32\|W\|(1+2\|W\|)} < \frac{1}{8\|W\|} .$$

Since $\dfrac{3 + 4\|W\|}{32\|W\|(1+2\|W\|)}$ is a monotonically decreasing function of the $\|W\|$ and $\|W\| \geqq 1$, $r_0 + \rho < \dfrac{7}{96} < \dfrac{1}{2}$. Thus if $\|u\|$, $\|y\| \leqq \rho$ then for $\|x_1\|$, $\|x_2\| \leqq r$ where $r_1 \leqq r \leqq r_0$,

$$\begin{aligned}\|G(x_2) - G(x_1)\| &= \|W(F(x_2 + u) - F(x_1 + u)\| \leqq \|W\| \, 8(r + \rho)\|x_2 - x_1\| \\ &= k(r)\|x_2 - x_1\|\end{aligned}$$

where $k(r) < 1$. Furthermore, we have that

$$\begin{aligned}\|G(0)\| = W(y + F(u)\| &\leqq \|W\|(\|y\| + 2\|u\|^2) \leqq \|W\|(\rho + 2\rho^2) \\ &\leqq r(1 - k(r)) = g(r).\end{aligned}$$

Consequently, $G(x)$ satisfies the hypothesis of Theorem 4.3 and the conclusion of the theorem follows.

We see that for $\|y\|, \|u\| < \rho < \dfrac{1}{16\|W\| + 32\|W\|^2}$ we have a unique function $x(y, u)$ such that $W(y + F(x(y, u) + u)) \equiv x(y, u)$.

Lemma 4.13: $x(y, u)$ is a continuous function for $\|y\|, \|u| < \rho$.

Proof: Let $\|y_1\|$, $\|u_1\|$, $\|y_2\|$, $\|u_2\|$ be less than ρ. We have

$$\begin{aligned}
&\| W(y_1 + F(x(y_2, u_2) + u_1)) - x(y_2, u_2) \| \\
&= \| W(y_1 + F(x(y_2, u_2) + u_1)) - W(y_2 + F(x(y_2, u_2) + u_2)) \| \\
&= \| W(y_1 - y_2) + W(F(x(y_2, u_2) + u_1) - F(x(y_2, u_2) + u_2)) \| \\
&\leqq \| W \| (\| y_1 - y_2 \| + 8(r_0 + \rho) \| u_1 - u_2 \|) = \varepsilon .
\end{aligned}$$

Therefore

$$\| x(y_2, u_2) - x(y_1, u_1) \| \leqq \frac{\varepsilon}{1 - k(r_0)}$$

(Lemma 4.4) which implies that $x(y, u)$ is continuous.

Theorem 4.14: (see for example [10]) Let $L \in X^X$ and $\| L \| < 1$, then $(I - L)^{-1} = I + \sum_{n=1}^{\infty} L^n$ exists.

Theorem 4.15: $x(y, u)$ is an analytic function for $\| x \|, \| u \| < \rho$.

Proof: Let $\overline{G}(x, y, u) = W(y + F(x + u))$. Then for $\| x \| \leqq r_0$, and $\| u \| \leqq \rho < \dfrac{1}{16\|W\| + 32(\|W\|)^2}$ we have

$$\begin{aligned}
\| \overline{G}_x(x, y, u) \| &= \| W(F'(x + u)) \| \\
&\leqq \| W \| \, \| F'(x + u) \| \leqq 8 \|W\| \, \| x + u \| \leqq 8 \|W\| (\|x\| + \|u\|) \\
&\leqq \| W \| \, 8(r_0 + \rho) < 1 .
\end{aligned}$$

Thus we have $\| \overline{G}_x(x, y, u) \| < 1$ and $(I - \overline{G}_x)^{-1}$ exists. Recall that $x = G(x, y, u)$ defines $x(y, u)$. If $\Delta x = x(y + \lambda h, u) - x(y, u)$, then

$$\begin{aligned}
\Delta x &= \overline{G}_x(x, y, u) \Delta x + o(\| \Delta x \|) + \overline{G}_y(x, y, u,)(\lambda h) \\
&\quad + o(\| \lambda h \|) = \overline{G}_x(x, y, u) \Delta x + \overline{G}_y(x, y, u)(\lambda h) \\
&\quad + o(\| \lambda h \|)
\end{aligned}$$

since $x(y, u)$ is continuous. Therefore we have

$$\Delta x = (I - \overline{G}_x)^{-1} [(\overline{G}_y)(\lambda h) + o(\| \lambda h \|)] \quad \text{and} \quad \lim_{\lambda \to 0} \frac{\Delta x}{\lambda} = (I - \overline{G}_x) \overline{G}_y .$$

Consequently

$$x_y(y, u) = (I - \overline{G}_x)^{-1} (\overline{G}_y)$$

and similarly we have

$$x_u(y, u) = (I - \overline{G_x})^{-1} \overline{G_u}.$$

Thus x has Gateau partial derivatives and since x is continuous it has Fréchet partial derivatives which implies that x is an analytic function for $\|y\|, \|u\| < \rho$.

q.e.d.

We originally had

1) $x_1 = W(y + F(x_1 + u)) = \overline{G}(x_1, y, u) = G(x_1)$ and

1) $x^*_{ij}(y + F(x_1 + u)) = 0$ for $i = 1, \dots, k$.

This is equivalent to for $\|y\|, \|u\| < \rho < \dfrac{1}{16\|W\| + 32\|W\|^2}$.

1) $x_1 = x(y, u)$ and

2) $x^*_{i1}(y + F(x_1 + u)) = 0$ for $i = 1, \dots, k$.

Thus we have for $\|y\| < \rho$, if there exists a $u \in M_1$, $\|u\| < \rho$ such that $x^*_{i1}(y + F(x(y, u) + u)) = f_i(y, u) = 0$ for $i = 0, \dots, k$ then $x = x(y, u) + u$ is a solution of the functional equation $x - f(x) = y$, and if $x_1 + u$, $x_1 \in \mathfrak{X}$ and $u \in M_1$ is a solution, then $x^*_{i1}(y + F(x_1 + u)) = 0$ and if $\|y\|, \|u\| < \rho$, $x_1 = x(y, u)$.

So we see that $f_i, i = 1, \dots, k$ are the branching equations that y and u need to satisfy in order for a solution of $x - f(x) = y$ to exist. It would be desirable if our branching equations were independent of u. Therefore further analysis is indicated.

We have for $\|y\|, \|u\| < \rho$, $f_i(y, u) = 0$ $i = 0, \dots, k$ where $u = \sum_{i=1}^{k} \lambda_i x_{iv_i}$ and the f_i's are analytic functions. Thus

$$f_i(y, u) = \sum_{j=0}^{\infty} P_{ij}(y, u),$$

where P_{ij} is homogeneous in u of j^{th} degree and analytic. Let v_i be the smallest j such that $P_{ij}(0, u) \not\equiv 0$. Note that v_i exists if and only if $f_i(0, u) \not\equiv 0$. We consider $P_{iv_i}(0, u)$ as a function on C_k to C. For $i = 1, \dots, k$, $\{u \in M_1 \mid P_{iv_i}(0, u) = 0\}$ is a $k - 1$ dimensional algebraic manifold in M_1. Therefore there exists a $u_0 \in M_1$ such that $P_{iv_i}(0, u_0) \neq 0$ for $i = 1, \dots, k$. (This result can be obtained using the concept of category.) We choose a basis of M_1 of the form $u_0, x_2, \dots, x_k$, and for

$$u \in M_1 \, , \; u = \lambda u_0 + \lambda_2 x_2 + \dots + \lambda_k x_k \, ,$$

where $\lambda, \lambda_2, \dots, \lambda_k \in C$. We have

$$P_{iv_i}(y, u) = \sum_{\Sigma \alpha_j \; v_i} a_{\alpha_1, \dots, \alpha_k}(y) \lambda^{\alpha_1} \lambda_2^{\alpha_2} \dots \lambda_k^{\alpha_k} \, ,$$

and

$$f_i(y, u_0) = a_{v_i, 0, \dots, 0}(y) \lambda^{v_i} + \dots ; \qquad a_{v_i, \dots, 0} \neq 0 \, .$$

Thus Theorem 2.4 implies that for $i = 1, \dots, k$

$$f_i(y, u) = \left(\lambda^{v_i} - \sum_{j=0}^{v_i - 1} H_{ij}(y, \lambda_2, \dots, \lambda_k) \lambda^j \right) \Omega_i(y, u)$$

where $H_{ij}(0, \dots, 0) = 0$ and H_{ij}, Ω_i are analytic in a neighborhood of the origin. Consequently $f_i(y, u) = 0$ if and only if

$$\lambda^{v_i} - \sum_{j=0}^{v_i - 1} H_{ij}(y, \lambda_2, \dots, \lambda_k) \lambda^j = g_i(\lambda) = 0 \, .$$

Definition 4.16: Let

$$h_1(\lambda) = a_0 \lambda^r + \dots + a_r$$

and

$$h_2(\lambda) = b_0 \lambda^s + \dots + b_s$$

be two polynomials, then we define

$$R(h_1, h_2) = \begin{vmatrix} a_0, & a_1, & a_2, & \dots, & a_r, & 0, & \dots & , 0 \\ 0, & a_0, & a_1, & \dots, & a_{r-1}, & a_r, & 0, \dots & , 0 \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ 0, & - & - & - & 0, & a_0, & a_1, \dots & , a_r \\ b_0, & b_1, & \dots, & b_s, & 0, & \dots & \dots & , 0 \\ 0, & b_0, & \dots, & b_{s-1}, & b_s, & \dots & \dots & , 0 \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ 0, & - & - & - & - & b_0, & b_1, \dots & , b_s \end{vmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \\ \\ \end{matrix}\right\} s \text{ rows} \\ \left.\begin{matrix} \\ \\ \\ \\ \\ \end{matrix}\right\} r \text{ rows} \end{matrix}$$

Theorem 4.17: (see Van der Waerden [16]). Given k

polynomials, $g_i(\lambda)$, $i=1, \ldots, k$, let

$$\varphi_1(\lambda) = \sum_{i=1}^{k} w_i g_i(\lambda) \quad \text{and} \quad \varphi_2(\lambda) = \sum_{i=1}^{k} v_i g_i(\lambda)$$

where w_i, v_i, $i=1, \ldots, k$ are indeterminates. Then $R(\varphi_1, \varphi_2)$ is a polynomial in the indeterminates $w_i, v_i, i=1, \ldots, k$. The coefficient of $w_1^{i_1} w_2^{i_2} \ldots w_k^{i_k} v_1^{j_1} \ldots v_k^{j_k}$ is a polynomial say d_s, in the coefficients of the g_i's. Then $g_i(\lambda)$, $i=1, \ldots, k$ have a common root if and only if $d_s=0$, $s=1, \ldots, m$.

Applying this result to our problem we obtain λ is a common root of

$$\lambda^{u_i} - \sum_{j=0}^{u_i-1} H_{ij}(y, \lambda_2, \ldots, \lambda_k)\lambda^j, \quad i=1, \ldots, k$$

if and only if $d_s(H_{ij}(y_2, \lambda_2, \ldots, \lambda_k))=0$ for $s=1, \ldots, m$. Since d_s is a polynomial in the H_{ij}'s, we have $d_s(H_{ij})$ are analytic functions from $X \times C_{k-1}$ to C.

Continuing in this manner, we arrive at a finite number of branching equations $q_j(y)$, $j=1, \ldots, t$ such that if $q_j(y)=0$, $j=1, \ldots, t$ then there exists a finite number of solutions to the equation $x-f(x)=y$.

§ 5. Unique Factorization of Monic Polynomials

In this section we consider another approach to the analysis of the equations $g_i(\lambda)=0$. We consider the coefficients of these polynomials in λ as elements of some ring R and ask whether there exists unique factorization in $R[\lambda]$. Observe that the H_{ij}'s are functions from a complex Banach space to the complex numbers C, and these functions are analytic in a neighborhood of the origin.

Definition 5.1: Let Y be a complex Banach space and R be the set of functions from Y to C, which are analytic at the origin, with the following identification: $f=g$ if and only if there exists a set A of 2nd category with respect to the origin such that $f \equiv g$ on A. The identity theorem states that $f \equiv g$ on its domain of analyticity. $f, g \in R$, then $(f+g)(x)=f(x)+g(x)$ and $(f \cdot g)(x)=f(x) \cdot g(x)$.

One easily proves that

1) R is a commutative ring with unit
2) R is an integral domain
3) if $f \in R$, f is a unit if and only if $f(0) \neq 0$
4) The non-units form an ideal.

Let Q be the quotient field of R, and $R[y]$, $Q[y]$ the polynomial rings of R and Q respectively. A polynomial in $R[y]$ whose coefficient of the highest power of y is a unit in R is called a monic polynomial.

Definition 5.2: Let J be an integral domain with unit, F its quotient field and $J[y]$ the polynomial ring of J. We say that J is integrally closed if for every monic polynomial $f(y)$ in $J(y)$, any root y_0 of f in F is in J.

Theorem 5.3: -(Butts, Hall and Mann [4]). Let J be an integrally closed integral domain with unit element and F its quotient field. Let $f(y) \in J[y]$ and $f(y) = g(y)\, h(y)$ where $g(y), h(y) \in F(y)$. Let f, g and h have first coefficients a, b, c, respectively. Then $\frac{a}{b}\, g(y)$, $\frac{a}{c}\, h(y)$ have integral coefficients and thus $af(y) = \left(\frac{a}{b}\, g(y)\right) \left(\frac{a}{c}\, h(y)\right)$ is a decomposition of $f(y)$ in $J[y]$.

Corollary 5.4: If J is integrally closed and the monic polynomial $f(y) \in J[y]$ factors in $F[y]$, then it factors in $J[y]$. Thus, monic polynomials have unique factorization.

Theorem 5.5: R is integrally closed

Proof: Let $f \in Q$, where f is a root of a monic polynomial: that is $f = \frac{g}{h}$, where $g, h \in R$ and

$$f^n + \sum_{j=0}^{n-1} a_j f^j = 0 \quad \text{where} \quad a_j \in R.$$

Since $h(x) \not\equiv 0$, there exists a x_0 such that $h(x_0) \neq 0$. We have $h(\lambda x_0) \not\equiv 0$, and $h(0) = 0$. Thus there exists an r such that for $|\lambda| = r$, $h(\lambda x_0) \neq 0$. Since $\{\lambda \in C \mid |\lambda| = r\}$ is a compact set, there is a neighborhood of the origin $N(0)$ such that if $y \in N(0)$ and $|\lambda| = r$, $h(y + \lambda x_0) \neq 0$.

If $P(z) = z^n + \sum_{j=0}^{n-1} a_j z^j$, then all the zeros of P are in the circle $|z| \leqq F(|a_0|, \ldots, |a_{n-1}|)$, where F is some function: for example if $|a_j| \leqq M$, $j = 1, \ldots, n-1$, $P(z) = 0$, then $|z| \leqq \max(nM, (nM)^{1/n})$.

For a fixed $y \in N(0)$, $f(y + \lambda x_0)$ is meromorphic function of λ ($h(y + \lambda x_0) \not\equiv 0$). From the remark in the preceding paragraph we see it is bounded and thus $f(y + \lambda x_0)$ is analytic in λ for $|\lambda| \leqq r$. Therefore for $y \in N(0)$

$$f(y) = \frac{1}{2\pi i} \int_{|\lambda|=r} \frac{f(y + \lambda x_0)}{\lambda} d\lambda .$$

Thus f is analytic in $N(0)$, and $f \in R$.

q.e.d.

We note that monic polynomials factor into products of monic polynomials. The generalization of the Weierstrass Preparation Theorem gives rise to special types of monic polynomials, namely, the leading coefficient is a unit but all other coefficients are non-units. Since the non-units form an ideal one can easily prove by induction that these polynomials factor into polynomials of similar type.

BIBLIOGRAPHY

[1] Banach, S. Théorie des opérations linéaires. Warszawa, 1932.

[2] Bochner, S. and Martin, R. S. Several complex variables. Princeton University Press, 1948.

[3] Butts H., Hall M. Jr, and Mann H. B. "On integral closure". *Canadian J. Math.*, 6 (1654), 471—473.

[4] Cronin, J. "Branch points of solutions of equations in Banach space". *Trans. Amer. Math. Soc.* 69 (1950), 208—231.

[5] Cronin, J. "Analytic functional mappings". *Ann. of Math.* (2) 58 (1953), 175—181.

[6] Hildebrandt, T. H. and Graves, L. M. "Implicit functions and their differentials in general analysis". *Trans. Amer. Math. Soc.* 29 (1927), 127—153.

[7] Hille, E. and Phillips, R. S. Functional Analysis and Semigtoups, *Amer. Math. Soc. Colloq. Pub.* vol. 31, revjsed edition, New York (1957).

[8] Lichtenstein, L. Vorlesungen über einige Klassen nichtlinearer Integralgleichungen und Integro-differentialgleichungen. J. Springer, Berlin, 1931.

[9] Lyapunov, A., Sur les figures d'équilibre peu différentes des ellipsoides d'une masse liquide homogène douée d'un mouvement de rotation I, étude générale du problème. *Acad. Imp. Sci.* St. Petersbourg, 1906.

[10] R i e s z, F. and S z - N a g y, B., Leçons d'analyse fonctionnelle, Académiai Kiado, Budapest, 1952.

[11] S c h m i d t, E., "Zur Theorie der linearen und nichtlinearen Integralgleichungen III. Ueber die Auflösung der nichtlinearen Integralgleichung und die Verzweigung iher Lösungen". *Math. Ann.* 65 (1808), 370—399.

[12] T a y l o r, A. E., Analytic functions and general analysis. Unpublished Dissertation, *Cal. Inst. of Tech.*, 1936.

[13] T a y l o r, A. E., "Analytic functions in general analysis". *Annali Della R. Scuola Normale Superiore di Pisa*, 6 (1937), 1—16.

[14] T a y l o r, A. E., "On the properties of analytic functions in abstract spaces". *Math. Annalen*, 115 (1938), 574—593.

[15] T i t c h m a r s h. E. C., The theory of functions (2nd ed). Oxford Univ. Press, 1939.

[16] V a n d e r W a e r d e n. B. L., Moderne algebra II. J. Springer, Berlin, 1931.

[17] Z o r n, M. A., "Characterization of analytic functions in Banach spaces". *Annals of Math.* (2) 46 (1945), 585—593.

[18] Z o r n, M. A., "Derivatives and Fréchet differentials". *Bull. Amer. Math. Soc.* 52 (1946), 133—137.

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(Received June 23, 1960)

AN ANALYTIC APPROACH TO FINITE FLUCTUATION PROBLEMS IN PROBABILITY (*)

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Introduction and Definitions

During the past 15 years a great deal of research has been done on finite fluctuation problems in probability theory. Basically, the problem is to calculate the distribution of random variables defined in terms of partial sums S_n of a sequence $\{X_k\}$ of independent and identically distributed random variables. One feature of this work is that emphasis is placed on finding the exact distribution of variables defined by a finite number of the S_n as opposed to finding limit theorems involving distributions of S_n for large n. Our purpose here is to show that one relatively simple method will give all the interesting results on finite fluctuation problems which have thus far been discovered. This method also gives a number of interesting new results concerning questions not previously considered.

It is amusing to note that the real interest in finite fluctuation problems was inspired by a limit theorem of Erdös and Kac [11]. In 1946 they proved the following result: Let $X_1, X_2, \ldots$ be independent random variables each having mean 0 and variance 1 and such that the central limit theorem is applicable. Let $S_k = X_1 + \ldots + X_k$ and let N_n denote the number of S_k's, $1 \leqq k \leqq n$, which are positive. Then

$$\lim_{n\to\infty} P\left\{\frac{N_n}{n} < \alpha\right\} = \frac{2}{\pi} \arc\sin \alpha^{1/2}, \qquad (0 \leqq \alpha \leqq 1). \tag{0.1}$$

A short time after the appearance of the result (0.1), E. Sparre Andersen [1] showed that the same result holds if $X_1, X_2, \ldots$ are independent and have a common symmetric and continuous distribution. In proving (0.1) Andersen presented the first significant result concerning finite fluctuation problems. He proved that, if $X_1, X_2, \ldots$ are independent and have a common continuous and symmetric distribution,

* Prepared under the auspices of National Science Foundation Grant G—9669.

$$(0.2) \qquad P\{N_n = m\} = (-1)^n \binom{-\frac{1}{2}}{m} \binom{-\frac{1}{2}}{n-m}.$$

Still later, Andersen [2,3] extended his own result (0.2) to the case in which the X_k's are neither symmetric nor have continuous distributions. He showed that $P\{N_n = m\}$ depends only on $P\{S_k > 0\}$ ($k = 1, 2, \ldots, n$) according to the identity

$$(0.3) \qquad \sum_{n=0}^{\infty} \lambda^n \sum_{m=0}^{n} U^m P\{N_n = m\} = \exp\left[\sum_{k=1}^{\infty} \frac{\lambda^k}{k} (U^k P\{S_k > 0\} + P\{S_k \leqq 0\})\right].$$

The importance of the form of this identity was further demonstrated by Andersen [4,5] in other examples.

The next significant contribution to the theory of finite fluctuation problems was made independently by Pollaczek [16] and by Spitzer [18]. Spitzer showed that the distribution of $M_n = \max(S_0, S_1, \ldots, S_n)$, where $S_0 = 0$, depends only on the distribution of S_k ($k = 1, 2, \ldots, n$) according to the identity

$$(0.4) \qquad \sum_{n=0}^{\infty} \lambda^n \int e^{itM_n}\, dP = \exp\left[\sum_{k=1}^{\infty} \frac{\lambda^k}{k} \left(\int_{\{S_k > 0\}} e^{itS_k}\, dP + P\{S_k \leqq 0\}\right)\right].$$

Pollaczek had found earlier essentially the same identity (0.4). The discovery by Spitzer of the identity (0.4) inspired a great deal of work on finite fluctuation problems by a number of people. We will mention here as a final result of interest one further identity presented by Wendel [22] in 1960. Let $S_0, S_1, \ldots, S_n$ be ordered according to decreasing value, and denote these ordered sums by $R_{n0} \geqq R_{n1} \geqq \ldots \geqq R_{nn}$. Wendel's result states that

$$(0.5) \qquad \sum_{n=0}^{\infty} \lambda^n \sum_{m=0}^{n} U^m \int e^{itR_{nm}}\, dP$$

$$= \exp\left[\sum_{k=1}^{\infty} \frac{\lambda^k}{k} \left(\int_{\{S_k > 0\}} e^{itS_k}\, dP + U^k P\{S_k > 0\} \right.\right.$$

$$\left.\left. + U^k \int_{\{S_k \leqq 0\}} e^{itS_k}\, dP + P\{S_k \leqq 0\}\right)\right].$$

We are now in a position to make two observations. First, the results stated in (0.3), (0.4) and (0.5) are all examples of one basic functional identity. In Part I of this paper we establish a method by which the above identities (and others) may be derived. The virtue of the method lies in its simplicity and its applicability. Our approach is basically analytical in contrast with the combinatorial approach of Andersen and Spitzer and Feller [13], and the algebraic treatment of Wendel. Next, we observe that none of the results stated above deals with joint distributions. In Part II of this paper we consider the problem of joint distributions using methods similar to those developed in Part I. Certain mathematical difficulties require us to restrict our attention in Part II to variables X_k which take on only integral values. For this restricted case we find results which are very similar to those listed in (0.3), (0.4) and (0.5).

Before stating a typical result of Part II, let us note a curious fact. In each identity (0.3), (0.4) and (0.5), the right hand side is a product of two types of functions. If we set

$$f_+(\lambda\,,t) = \exp\left[\sum_{k=1}^{\infty}\frac{\lambda^k}{k}\int_{\{S_k>0\}} e^{itS_k}\,dP\right] \tag{0.6}$$

and

$$f_-(\lambda\,,t) = \exp\left[\sum_{k=1}^{\infty}\frac{\lambda^k}{k}\int_{\{S_k\leqq 0\}} e^{itS_k}\,dP\right], \tag{0.7}$$

then Spitzer's identity (0.4), for example, can be written

$$\sum_{n=0}^{\infty}\lambda^n\int e^{itM_n}\,dP = f_+(\lambda\,,t)\,f_-(\lambda\,,0)\,. \tag{0.4}$$

Similar expressions hold for (0.3) and (0.5).

The results of Part II will show that product formulas like (0.4) hold also for joint distributions involving the "range"

$$R_n = \max(S_0\,,S_1\,,\dots,S_n) - \min(S_0\,,S_1\,,\dots,S_n)\,.$$

Specifically, let $X_1\,,X_2\,,\dots$ be independent and identically distributed lattice variables $(X_k = 0\,,\ \pm 1\,,\pm 2,\dots)$ and let $A_k = \delta_{k0} - \lambda P\{X_1 = k\}$. Let $D_n = \det(A_{j-i})\,,\ (i\,,j = 0,\,1,\dots,n)\,,$ let

$$(0.8)\qquad f_n^+(\lambda, t) = \frac{1}{D_{n-1}} \begin{vmatrix} A_0 & A_1 & \cdots & A_n \\ A_{-1} & A_0 & \cdots & A_{n-1} \\ \vdots & \vdots & & \vdots \\ A_{-n+1} & A_{-n+2} & \cdots & A_1 \\ z^n & z^{n-1} & \cdots & 1 \end{vmatrix}, \qquad (z = e^{it}),$$

and let

$$(0.9)\qquad f_n^-(\lambda, t) = \frac{1}{D_n} \begin{vmatrix} A_0 & A_1 & \cdots & A_{n-1} & z^{-n} \\ A_{-1} & A_0 & \cdots & A_{n-2} & z^{-n+1} \\ \vdots & \vdots & & \vdots & \vdots \\ A_{-n} & A_{-n+1} & \cdots & A_{-1} & 1 \end{vmatrix}, \qquad (z = e^{it}),$$

Our analogue of (0.4) is then

$$(0.10)\qquad \sum_{k=0}^{\infty} \lambda^k \int_{\{R_k \leqq 0\}} e^{itM_k}\, dP = f_n^+(\lambda, t)\, f_n^-(\lambda, t).$$

Of course, the joint distribution of $\max(S_0, S_1, \ldots, S_n)$ and $\min(S_0, S_1, \ldots, S_n)$ can be calculated from the result (0.10).

The polynomials $f_n^+(\lambda, t)$ and $f_n^-(\lambda, t)$ defined in (0.8) and (0.9) have an extremely important biorthogonality relation which is crucial in our methods. If we let $\varphi(t)$ denote the characteristic function of X_1 (lattice variable), then it will be shown that

$$(0.11)\qquad \frac{1}{2\pi} \int_{-\pi}^{\pi} [e^{int} f_n^-(\lambda, t)]\, [e^{-int} f_m^+(\lambda, t)]\, [1 - \lambda\varphi(t)]\, dt = \delta_{nm}.$$

Perhaps even more important is the observation that (0.11) implies that $f_n^+(\lambda, t)\, [1 - \lambda\varphi(t)]$ is a Fourier series with zero coefficients of e^{ikt} for $k = 1, 2, \ldots, n$. As we will see later, this last fact connects very closely the functions $f_+(\lambda, t)$ and $f_n^+(\lambda, t)$.

In Section 4 of Part I we apply our method to the problem of change of sign. Although limiting distributions for changes of sign are known [9], (*) and although distributions for the number of changes in sign among $S_0, S_1, \ldots, S_n$ have been given for special variables X_k [12], this problem is generally conspicuous in its absence from papers on finite fluctuation problems. There appear to be no "invariant" results of the type (0.2) for

* See also Darling and Kac, Trans. Amer. Math. Soc. 84, p. 455.

the number of changes of sign. It is hoped that the discussion of Section 4 Part I will shed some light on the essentially more difficult character of this problem. The results presented in Section 4 are in large part joint work of the author with E. Sparre Andersen.

We now list for future reference the definitions of variables which will be used in the sequel. In all cases $S_0, S_1, \dots, S_n$ will denote the partial sums of a sequence of independent and identically distributed random variables $\{X_k\}$.

N_n: the number of positive partial sums among $S_0, S_1, \dots, S_n$

L_n : the first index k such that $S_k = \max(S_0, S_1, \dots, S_n)$

$\overline{L}_n$: the first index k such that $S_k = \min(S_0, S_1, \dots, S_n)$

$M_n = \max(S_0, S_1, \dots, S_n)$

$\overline{M}_n = \min(S_0, S_1, \dots, S_n)$

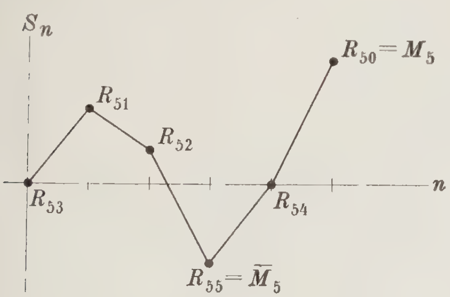
$R_n = \max(S_0, S_1, \dots, S_n) - \min(S_0, S_1, \dots, S_n)$

(0.12) C_n : the number of indices k such that either $S_{k-1} > 0$ and $S_k \leqq 0$, or $S_{k-1} \leqq 0$ and $S_k > 0$

$R_{n,k}$: the partial sums $S_0, S_1, \dots, S_n$ are ordered according to decreasing value and the ordered sums are denoted by $R_{n,0} \geqq R_{n,1} \geqq \dots \geqq R_{n,n}$

$L_{n,k}$: the index m such that $S_m = R_{n,k}$.

In order that the index variables $L_{n,k}$ be well defined we need to make a convention. If several partial sums among $S_0, S_1, \dots, S_n$ have the same value, we will order these partial sums according to increasing value of the subscripts. This will always give a unique ordering of the partial sums. In the sequel it will be convenient at times to analyze a simple picture called a "path". A path is just a plot (with connecting lines) of p o s s i b l e values of $S_0, S_1, \dots, S_n$. Such a path is drawn below to illustrate the variables defined above.



$N_5 = 3$, $L_5 = 5$, $\overline{L}_5 = 3$, $C_5 = 3$

$L_{50} = 5$, $L_{51} = 1$, $L_{52} = 2$, $L_{53} = 0$, $L_{54} = 4$, $L_{55} = 3$

Fig. 1.

PART I

1. A factorization problem:

In this section we will find the distribution of the variables L_n and M_n by a procedure which brings clearly into focus the central mathematical problem of the method of Part I. The procedure used in this section is closely related to the Wiener-Hopf factorization technique of classical analysis. We point out, however, that it is more general than the Wiener-Hopf technique because of special properties of the functions with which we are dealing.

In all we will derive four identities similar to those listed in (0.3), (0.4) and (0.5). In the notation of (0.6) and (0.7) these identities are as follows:

Identities:

i) (Andersen [4])

$$\sum_{n=0}^{\infty} \lambda^n \sum_{m=0}^{n} U^m P\{L_n = m\} = f_+(\lambda U, 0)\, f_-(\lambda, 0),$$

(1.1)

ii) $$\sum_{n=0}^{\infty} \lambda^n \int_{\{L_n=n\}} e^{itS_n}\, dP = f_+(\lambda, t),$$

iii) $$\sum_{n=0}^{\infty} \lambda^n \int_{\{L_n=0\}} e^{itS_n}\, dP = f_-(\lambda, t),$$

iv) (Spitzer [18])

$$\sum_{n=0}^{\infty} \lambda^n \int e^{itM_n}\, dP = f_+(\lambda, t)\, f_-(\lambda, 0).$$

It will be most convenient to prove ii) and iii) first.

Proof: Step I: For notational convenience let

$$p_n(t) = \int_{\{L_n=n\}} e^{itS_n}\, dP \quad \text{and} \quad q_n(t) = \int_{\{L_n=0\}} e^{itS_n}\, dP. \tag{1.2}$$

If $\varphi(t)$ denotes the characteristic function of X_1, then

$$\varphi^n(t) = \int e^{itS_n}\, dP = \sum_{k=0}^{n} \int_{\{L_n=k\}} e^{itS_n}\, dP. \tag{1.3}$$

A typical "path" satisfying the condition $L_n = k$ is illustrated below for the case $k = 2$ and $n = 5$. It is easy to see that

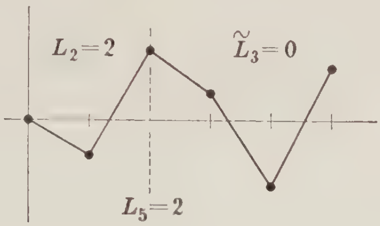


Fig. 2

such a path can be described by separate conditions on the sets of variables $X_1, \dots, X_k$ and $X_{k+1}, \dots, X_n$. If we let $\tilde{L}_{n-k}$ denote the variable "L" applied to $Y_1 = X_{k+1}, \dots, Y_{n-k} = X_n$, then $\{L_n = k\} = \{L_k = k\} \cap \{\tilde{L}_{n-k} = 0\}$. Thus, using the independence and identical distribution property of the X_k's we may write

$$\int_{\{L_n=k\}} e^{itS_n}\, dP = \int_{\{L_k=k\}\cap\{\tilde{L}_{n-k}=0\}} e^{itS_k} \cdot e^{it(S_n-S_k)}\, dP = p_k(t)\, q_{n-k}(t). \tag{1.4}$$

From (1.3) and (1.4) we get that for all $n \geqq 0$

$$\varphi^n(t) = \sum_{k=0}^{n} p_k(t)\, q_{n-k}(t). \tag{1.5a}$$

In terms of the generating functions $P(\lambda, t)$ and $Q(\lambda, t)$ of $p_k(t)$ and $q_k(t)$ relation (1.5) gives o n e equation in the t w o unknown functions $P(\lambda, t)$ and $Q(\lambda, t)$:

$$\frac{1}{1-\lambda\varphi} = P(\lambda, t)\, Q(\lambda, t). \tag{1.5b}$$

Step II : We prove next a lemma related to Wiener-Hopf factorization which shows that (1.5) is an "algebraic" condition which uniquely determines

both $p_k(t)$ and $q_k(t)$. D. Ray [17] seems to have been the first person to use a Wiener-Hopf factorization procedure in fluctuation problems.

Lemma 1.1: Let $\varphi(t)$ be a known Fourier-Stieltjes transform of a function of bounded variation and let $\{p_k(t)\}$ and $\{q_k(t)\}$ be sequences of functions which satisfy

(1) $p_0(t) = q_0(t) = 1$

(2) $$p_k(t) = \int_{0+}^{\infty} e^{itx}\, dG_k(x) \qquad (G_k(x) \text{ of B.V.})$$

(3) $$q_k(t) = \int_{-\infty}^{0+} e^{itx}\, dG_k(x) \qquad (G_k(x) \text{ of B.V.})$$

(4) $$\varphi^n(t) = \sum_{k=0}^{n} p_k(t)\, q_{n-k}(t).$$

Then, $p_k(t)$ and $q_k(t)$ are uniquely determined for all k.

Proof: Assume that $q_m(t)$ and $p_m(t)$ have been determined for all $m < k$. From (4) with $n = k$

$$\varphi^k(t) = p_k(t) + q_k(t) + \sum_{m=1}^{k-1} p_m(t)\, q_{k-m}(t). \tag{1.6}$$

Thus, by the induction hypothesis

$$p_k(t) + q_k(t) = \int_{-\infty}^{\infty} e^{itx}\, dK(x), \tag{1.7}$$

where $K(x)$ is known. Properties (2) and (3) together with the uniqueness theorem for Fourier-Stieltjes transforms implies

$$p_k(t) = \int_{0+}^{\infty} e^{itx}\, dK(x) \quad \text{and} \quad q_k(t) = \int_{-\infty}^{0+} e^{itx}\, dK(x). \tag{1.8}$$

We note in passing that the lemma would remain true if the ranges of integration in (2) and (3) were replaced by A and A' for any linear Borel set A.

Step III: We have left only to demonstrate that $f_+(\lambda, t)$ and $f_-(\lambda, t)$ are generating functions of quantities which satisfy the conditions of Lemma 1.1 and identities ii) and iii) will follow. By (0.6) and (0.7) we have $1/[1-\lambda\varphi(t)] = f_+(\lambda, t)\, f_-(\lambda, t)$, showing condition (4) is satisfied. To see that (1), (2) and (3) are also satisfied it suffices to observe that sums and products of terms

$$(1.9) \qquad \int_{0+}^{\infty} e^{itx}\, dG(x) \quad \left(\text{or} \int_{-\infty}^{0+} e^{itx}\, dG(x)\right)$$

are again terms of the same type. Since $\log f_+(\lambda, t)$ (or $\log f_-(\lambda, t)$) is of type (1.9), we are finished with the proof.

To prove identities i) and iv) we merely repeat the first step of the proof just given.

Proof of iv): Consider

$$(1.10) \qquad \int e^{itM_n}\, dP = \sum_{k=0}^{n} \int_{\{L_n=k\}} e^{itM_n}\, dP = \sum_{k=0}^{n} \int_{\{L_k=k\}\cap\{\tilde{L}_{n-k}=0\}} e^{itS_k}\, dP$$
$$= \sum_{k=0}^{n} p_k(t)\, q_{n-k}(0)\,.$$

Identity iv) follows from (1.10) by taking generating functions of both sides.

Proof of i): Consider

$$(1.11) \qquad \sum_{k=0}^{n} U^k \int_{\{L_n=k\}} 1\cdot dP = \sum_{k=0}^{n} U^k \int_{\{L_k=k\}\cap\{\tilde{L}_{n-k}=0\}} 1\cdot dP$$
$$= \sum_{k=0}^{n} U^k p_k(0)\, q_{n-k}(0)\,.$$

Clearly, (1.11) is equivalent to identity i).

The most important part of the proof just given from the viewpoint of future considerations is the demonstration of an explicit pair of functions $f_+(\lambda, t)$ and $f_-(\lambda, t)$ (with obvious properties) satisfying

$$(1.12) \qquad f_+(\lambda, t)\, f_-(\lambda, t) = 1/[1-\lambda\varphi(t)]\,.$$

The construction of these explicit functions is one of the central mathematical

problems of our method of the next section. As we will see, it is not always possible to find "factors" explicitly. The crux of our future methods is a straightforward generalization of Lemma 1.1 in which condition (1.5b) is replaced by

$$P(\lambda, t)\,[1 - \lambda\varphi(t)] = Q(\lambda, t). \tag{1.13}$$

2. An operator:

The distribution of each variable listed in (0.12) is theoretically determined as soon as the characteristic function $\varphi(t)$ of X_1 is specified. It is therefore of some interest to see if the results of the previous section can be expressed explicitly in terms of $\varphi(t)$. To accomplish this end we introduce a notation.

Notation: For any function

$$\psi = \int_{-\infty}^{\infty} e^{itx}\, dH(x), \qquad (H(x) \text{ is of B.V.}), \tag{2.1}$$

let

$$\psi^{+} \equiv \int_{0+}^{\infty} e^{itx}\, dH(x) \quad \text{and} \quad \psi^{-} \equiv \int_{-\infty}^{0+} e^{itx}\, dH(x). \tag{2.2}$$

Supressing dependence on t, we can now write the expressions in (0.6) and (0.7) in the form

$$\begin{aligned} f_{+} &= \exp\left[\sum_{k=1}^{\infty} \frac{\lambda^k}{k} (\varphi^k)^{+}\right], \\ f_{-} &= \exp\left[\sum_{k=1}^{\infty} \frac{\lambda^k}{k} (\varphi^k)^{-}\right]. \end{aligned} \tag{2.3}$$

The identities can be now easily written in terms of φ, with, of course, the notations $+$ and $-$.

Actually, much more than a notation has been introduced. Relation (2.2) defines operators $+$ and $-$ which are fundamental to the theory of finite fluctuation problems. We will show how these operators lead to a simple and yet powerful method for analyzing any particular problem.

Our method is an expansion of one presented by the author in a previous paper [6]. The algebraic treatment of fluctuation problems used by Wendel [21] seems to have many features in common with our method.

There are three important properties of the operator $+$ defined by (2.2). Similar properties hold for the operator $-$.

Properties:

P1. Linear: $(a\psi_1 + b\psi_2)^+ = a\psi_1^+ + b\psi_1^+$, where a and b are complex numbers.

P2. Idempotent: $(\psi^+)^+ = \psi^+$.

P3. Closure: The class A^+ of all functions ψ^+, where ψ is of form (2.1) is closed under addition, multiplication, and multiplication by a complex constant.

We note in particular that $(\psi^-)^+ = 0 = (\psi^+)^-$ and that $\psi = \psi_1^+ + \psi_2^-$ implies $\psi_1^+ = \psi^+$ and $\psi_2^- = \psi^-$. The best way to explain the method of $+$ operators is to illustrate its use in a few simple examples. In all examples φ will denote the characteristic function of X_1.

Example 2.1: All positive sums. In our first example we consider in some detail the evaluation of $P\{N_n = n\}$. Let

$$\varphi_n = \int_{\{N_n=n\}} e^{itS_n}\, dP. \tag{2.4}$$

By independence of the X_k's

$$\varphi\varphi_n = \int_{\{N_n=n\}} e^{itS_n} \cdot e^{itX_{n+1}}\, dP = \int_{\{N_n=n\}} e^{itS_{n+1}}\, dP. \tag{2.5}$$

We have interpreted the product $\varphi\varphi_n$ as "adding a step to the end of a path which satisfies the condition $N_n = n$". The last integral in (2.5) is of

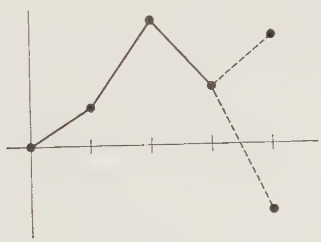


Fig. 3

the form (2.1) and we can apply the $+$ operator to it. The operator has the interpretation of eliminating all paths from the range of integration which have $S_{n+1} \leqq 0$. Thus,

$$(2.6) \qquad (\varphi\varphi_n)^+ = \int_{\{N_{n+1} \ n+1\}} e^{itS_{n+1}} \, dP = \varphi_{n+1}, \qquad (\varphi_0 = 1).$$

Letting

$$(2.7) \qquad \psi = \sum_{n=0}^{\infty} \lambda^n \varphi_n ,$$

we get from (2.7)

$$(2.8) \qquad \psi = 1 + \lambda(\varphi\psi)^+ = 1 + \lambda \sum_{n=0}^{\infty} \lambda^n (\varphi\varphi_n)^+ .$$

The interchange of the summation and $+$ operator in (2.8) can easily be justified. We note that there is a unique power series solution ψ to (2.8), since by equating coefficients of like powers of λ on both sides of (2.8) we are led back to the recurrence relation (2.6).

To find the solution of (2.8) we first note that (2.8) and P2. imply $(\psi-1)^+ = \psi - 1$. Since $1^+ = 0$, we have also that $\psi - 1 = (\psi-1)^+ = \psi^+$. Thus, we can rewrite (2.8) in the form

$$(2.9) \qquad [\psi(1-\lambda\varphi)]^+ = 0 \quad \text{and} \quad \psi^- = 1 .$$

But

$$(2.10) \qquad \psi = \exp\left[\sum_{k=1}^{\infty} \frac{\lambda^k}{k} (\varphi^k)^+\right]$$

is a function with $\psi^- = 1$ such that

$$\psi(1-\lambda\varphi) = \exp\left[-\sum_{k=1}^{\infty} \frac{\lambda^k}{k} (\varphi^k)^-\right] \in A^-, \qquad \text{(See P3.)}.$$

Thus, $[\psi(1-\lambda\varphi)]^+ = 0$, and the solution is given in (2.10). To find the generating function for $P\{N_n = n\}$ from ψ we merely set $t = 0$.

Example 2.2: F i r s t n o n - p o s i t i v e s u m a m o n g $S_1, S_2, \ldots, S_n$. Let

$$\psi_n = \int_{\begin{Bmatrix} N_{n-1}=n-1 \\ S_n \leqq 0 \end{Bmatrix}} e^{itS_n}\,dP, \qquad (n \geqq 1), \tag{2.11}$$

and let φ_n be given by (2.4). From (2.5) and an obvious interpretation of the $-$ operator, we get

$$(\varphi\varphi_n)^- = \int_{\begin{Bmatrix} N_n=n \\ S_{n+1} \leqq 0 \end{Bmatrix}} e^{itS_{n+1}}\,dP = \psi_{n+1}. \tag{2.12}$$

Equivalently,

$$\psi_{n+1} = \varphi\varphi_n - (\varphi\varphi_n)^+ = \varphi\varphi_n - \varphi_{n+1}. \tag{2.13}$$

In terms of generating functions, (2.13) states

$$\sum_{n=1}^{\infty} \lambda^n \psi_n = 1 - (1-\lambda\varphi)\,\psi = 1 - \exp\left[-\sum_{k=1}^{\infty} \frac{\lambda^k}{k}(\varphi^k)^-\right]. \tag{2.14}$$

The last equality in (2.14) follows from a use of (2.10).

Example 2.3: Maximum at endpoint. In this example we exhibit a slight but very useful variation of the argument of Example 2.1. Let

$$\varphi_n = \int_{\{L_n=n\}} e^{itS_n}\,dP. \tag{2.15}$$

Furthermore, let $\tilde{L}_n$ be defined for $Y_1 = X_2, \ldots, Y_n = X_{n+1}$ exactly as L_n is defined for $X_1, \ldots, X_n$. Then

$$\varphi\varphi_n = \int_{\{\tilde{L}_n=n\}} e^{itS_{n+1}}\,dP. \tag{2.16}$$

We have interpreted $\varphi\varphi_n$ as "adding a step at the beginning of a path satisfying $L_n = n$". The "new" path will satisfy the condition $L_{n+1} = n+1$ if, and only if, $S_{n+1} > 0$ (See Fig. 4). Thus,

$$(\varphi\varphi_n)^+ = \int_{\{L_{n+1}=n+1\}} e^{itS_{n+1}}\,dP = \varphi_{n+1}, \qquad (\varphi_0 = 1). \tag{2.17}$$

Fig. 4

Relations (2.17) and (2.6) are identical, showing that φ_n in (2.15) and φ_n in (2.4) are the same. Once again (2.10) gives the generating function for the expressions in (2.15).

Earlier we mentioned the dual properties P1.—P3. of the + and − operators. Assuming that there is some justice in mathematics, we should expect a dual for every result thus far found. We expect a dual of Example 2.1 concerning all non-negative sums of the form

$$\sum_{n=0}^{\infty} \lambda^n \int_{\{L_n=0\}} e^{itS_n}\, dP = \exp\left[\sum_{k=1}^{\infty} \frac{\lambda^k}{k} (\varphi^k)^-\right]. \tag{2.18}$$

Corresponding to Example 2.2, we expect that

$$\sum_{n=0}^{\infty} \lambda^n \int_{\left\{\substack{L_{n-1}=0 \\ S_n>0}\right\}} e^{itS_n}\, dP = 1 - \exp\left[-\sum_{k=1}^{\infty} \frac{\lambda^k}{k} (\varphi^k)^+\right]. \tag{2.19}$$

Fortunately, the results (2.18) and (2.19) are quite easy to prove using the methods described in the examples above, showing that there is some justice in mathematics after all.

3. Several major results:

In this section we will indicate the power of the method of ± operators by briefly describing how identities (0.3), (0.4) and (0.5) can be derived from this method. We also give two results which seem to have appeared previously only in special cases.

Example 3.1: Positive partial sums (Andersen [4]). Let

$$\varphi_n = \sum_{m=0}^{n} U^m \int_{\{N_n=m\}} e^{itS_n}\, dP, \qquad (\varphi_0 = 1). \tag{3.1}$$

Then,

$$\varphi\varphi_n = \sum_{m=0}^{n} U^m \int\limits_{\{N_n=m\}} e^{itS_{n+1}}\,dP\,.$$

Applying the operators we find

$$(3.2)\qquad U(\varphi\varphi_n)^+ = \sum_{m=0}^{n} U^{m+1} \int\limits_{\left\{\begin{smallmatrix} N_{n+1}=m+1 \\ S_{n+1}>0 \end{smallmatrix}\right\}} e^{itS_{n+1}}\,dP$$

$$= \sum_{m=1}^{n+1} U^m \int\limits_{\left\{\begin{smallmatrix} N_{n+1}=m \\ S_{n+1}>0 \end{smallmatrix}\right\}} e^{itS_{n+1}}\,dP\,,$$

and

$$(3.3)\qquad (\varphi\varphi_n)^- = \sum_{m=0}^{n} U^m \int\limits_{\left\{\begin{smallmatrix} N_{n+1}=m \\ S_{n+1}\leqq 0 \end{smallmatrix}\right\}} e^{itS_{n+1}}\,dP\,.$$

It is easily seen that equality still holds in (3.2) and (3.3) if the summation on the right is extended to $m = 0, 1, \ldots, n+1$. By adding,

$$(3.4)\qquad U(\varphi\varphi_n)^+ + (\varphi\varphi_n)^- = \varphi_{n+1}\,.$$

In terms of the generating function ψ of φ_n relation (3.4) states that

$$(3.5)\qquad \psi = 1 + \lambda U(\varphi\psi)^+ + \lambda(\varphi\psi)^-\,,$$

or equivalently,

$$(3.6)\qquad [\psi(1-\lambda U\varphi)]^+ + [\psi(1-\lambda\varphi) - 1]^- = 0\,.$$

Each term in (3.6) must necessarily vanish. If we set $P = \psi(1-\lambda\varphi)$ and $Q = \psi(1-\lambda U\varphi)$, we have $P - 1 \in A^+$, $Q \in A^-$, and

$$(3.7)\qquad \frac{P}{Q} = \frac{1-\lambda\varphi}{1-\lambda U\varphi}\,.$$

Thus, we are led to a "factorization" problem of the type discussed in Section 1. Using the unique factors

$$P = \exp\left[\sum_{k=1}^{\infty} \frac{\lambda^k}{k} \{U^k (\varphi^k)^+ - (\varphi^k)^+\}\right]$$

$$Q = \exp\left[-\sum_{k=1}^{\infty} \frac{\lambda^k}{k} \{U^k (\varphi^k)^- - (\varphi^k)^-\}\right]$$

of (3.7), we are led to the result

$$\psi = P/[1 - \lambda\varphi] = \exp\left[\sum_{k=1}^{\infty} \frac{\lambda^k}{k} \{U^k (\varphi^k)^+ + (\varphi^k)^-\}\right]. \tag{3.8}$$

The identity of (0.3) follows from (3.8) by setting $t = 0$.

Example 3.2: Distribution of max $(S_0, S_1, \ldots, S_n)$ (Spitzer [18]). Let

$$\varphi_n = \int e^{itM_n}\, dP \tag{3.9}$$

and let $\tilde{M}_n$ denote the maximum of the partial sums of $X_2, \ldots, X_{n+1}$ (including $S_0 = 0$). Then,

$$\varphi\varphi_n = \int e^{it(\tilde{M}_n + X_1)}\, dP\,. \tag{3.10}$$

But $\tilde{M}_n + X_1 = M_{n+1}$, if $M_{n+1} > 0$. Thus,

$$\begin{aligned}(\varphi\varphi_n)^+ = \int_{\{M_{n+1}>0\}} e^{itM_{n+1}}\, dP &= \varphi_{n+1} - P\{M_{n+1} = 0\} \\ &= \varphi_{n+1} - P\{N_{n+1} = 0\}\,.\end{aligned} \tag{3.11}$$

The generating function, say C, of $P\{N_n = 0\}$ is found by equating coefficients of U° (with $t = 0$) in (3.8), i.e.

$$C = \sum_{n=0}^{\infty} \lambda^n P\{N_n = 0\} = \exp\left[\sum_{k=1}^{\infty} \frac{\lambda^k}{k} P\{S_k \leqq 0\}\right]. \tag{3.12}$$

Thus, if ψ is the generating function of φ_n we get from (3.11)

$$\frac{\psi}{C} = 1 + \lambda\left(\varphi\,\frac{\psi}{C}\right)^+, \tag{3.13}$$

whose solution according to Example 2.1 is simply

$$\frac{\psi}{C} = \exp\left[\sum_{k=1}^{\infty} \frac{\lambda^k}{k} (\varphi^k)^+\right]. \tag{0.4}$$

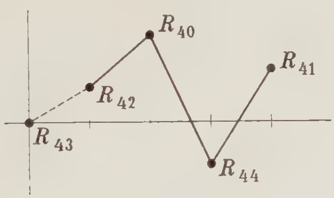
Example 3.3: Order Statistics (Wendel [22]). Let

$$\varphi_n = \sum_{k=0}^{n} U^k \int e^{itR_{nk}}\, dP \tag{3.14}$$

and let $\tilde{R}_{nk}$ be the order statistics for the partial sums of $X_2, \dots, X_{n+1}$. Then,

$$\varphi\varphi_n = \sum_{k=0}^{n} U^k \int e^{it(\tilde{R}_{nk}+X_1)}\, dP, \tag{3.15}$$

where a step has been added at the beginning of a path (see Fig. 5). The relative positions of the points on the "old" path are unchanged by the addition of a new point at the origin. Thus, $\tilde{R}_{nk} + X_1 = R_{n+1,k}$ if $R_{n+1,k} > 0$,



$$R_{42} = \tilde{R}_{32} + X_1$$

$$R_{44} = \tilde{R}_{33} + X_1$$

Fig. 5

and $\tilde{R}_{nk} + X_1 = R_{n+1,k+1}$ if $R_{n+1,k+1} \leqq 0$. In no case is $\tilde{R}_{nk} + X_1 \equiv S_0$. In particular, we note

$$\begin{aligned} U(\varphi\varphi_n)^- &= \sum_{k=0}^{n} U^{k+1} \int_{\{S_0 \not\equiv R_{n+1,k+1} \leqq 0\}} e^{itR_{n+1,k+1}}\, dP \\ &= \sum_{k=0}^{n+1} U^k \int_{\{R_{n+1,k} \leqq 0\}} e^{itR_{n+1,k}}\, dP - \sum_{k=0}^{n+1} U^k P\{R_{n+1,k} \equiv S_0\}. \end{aligned} \tag{3.16}$$

But, $R_{n+1,k} \equiv S_0$ if, and only if, $N_{n+1} = k$. Thus,

$$(\varphi\varphi_n)^+ + U(\varphi\varphi_n)^- = \varphi_{n+1} - \sum_{k=0}^{n+1} U^k P\{N_{n+1} = k\}. \tag{3.17}$$

If K is the generating function given in (0.3) and ψ is the generating function of φ_n, then (3.17) gives

$$\frac{\psi}{K} = 1 + \lambda\left(\varphi \frac{\psi}{K}\right)^{+} + \lambda U \left(\varphi \frac{\psi}{K}\right)^{-}. \tag{3.18}$$

This last equation is the "dual" of (3.5), and we can therefore immediately write from (3.8)

$$\psi = K \exp\left[\sum_{k=1}^{\infty} \frac{\lambda^k}{k} \{(\varphi^k)^{+} + U^k (\varphi^k)^{-}\}\right]. \tag{0.5}$$

Finally, we state without proof two generalizations of examples previously considered. These identities appear to be new, altough special cases have appeared previously.

Example 3.4: Order indices. Let

$$\varphi_n = \sum_{k, m=0}^{n} V^k U^m \int_{\{L_{nk}=m\}} e^{itR_{nk}}\, dP. \tag{3.19}$$

Then, the generating function ψ of φ_n in terms of the notation (0.6) and (0.7) is

$$\psi = f_{+}(\lambda U, t)\, f_{+}(\lambda V, 0)\, f_{-}(\lambda U V, t)\, f_{-}(\lambda, 0). \tag{3.20}$$

Setting $t = 0$ in (3.20) gives a result of Darling [10] on order indices.

Example 3.5: Distribution of kth positive sum. Let

$$\varphi_n = \sum_{k=1}^{n} U^k \int_{\substack{\{N_n = k\} \\ \{S_n > 0\}}} e^{itS_n}\, dP, \qquad (n \geq 1). \tag{3.21}$$

Then, the generating function of φ_n is

$$\psi = \frac{U}{1-U}\left[1 - \exp\left\{\sum_{k=1}^{\infty} \frac{\lambda^k}{k} (U^k - 1)(\varphi^k)^{+}\right\}\right]. \tag{3.22}$$

Setting $t = 0$ in (3.22) gives results of Andersen [5].

4. Change of sign:

We now apply our method to a much more difficult problem than those considered up to now. It is hoped that the discussion of this section

will provide a deeper insight into the real character of the previous identities as well as explain why the change of sign is basically a more difficult problem. A number of the ideas presented here were conceived in joint work with E. Sparre Andersen.

To analyze change of sign we consider two functions:

$$\varphi_n = \sum_{k=0}^{n} U^k \int_{\substack{C_n = k \\ S_n > 0}} e^{itS_n}\, dP\,, \qquad \left(\Phi = \sum_0^\infty \lambda^n \varphi_n\right), \tag{4.1}$$

and

$$\psi_n = \sum_{k=0}^{n} U^k \int_{\substack{C_n = k \\ S_n \leqq 0}} e^{itS_n}\, dP\,, \qquad \left(\Psi = \sum_0^\infty \lambda^n \psi_n\right). \tag{4.2}$$

It follows that

$$(\varphi\varphi_n)^+ = \sum_{k=0}^{n} U^k \int_{\substack{C_{n+1} = k \\ S_n > 0 \\ S_{n+1} > 0}} e^{itS_{n+1}}\, dP \tag{4.3}$$

and

$$U\,(\varphi\psi_n)^+ = \sum_{k=1}^{n+1} U^k \int_{\substack{C_{n+1} = k \\ S_n \leqq 0 \\ S_{n+1} > 0}} e^{itS_{n+1}}\, dP\,. \tag{4.4}$$

From (4.3) and (4.4) and a similar argument with the — operator, we find

$$\begin{cases} (\varphi\varphi_n)^+ + U\,(\varphi\psi_n)^+ = \varphi_{n+1} & (\varphi_0 = 0) \\ U\,(\varphi\varphi_n)^- + (\varphi\psi_n)^- = \psi_{n+1} & (\psi_0 = 1)\,. \end{cases} \tag{4.5}$$

The recurrence relation (4.5) implies an obvious pair of equations for the generating functions Φ and Ψ. It will be convenient to write these equations in matrix form:

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \left[\begin{pmatrix} 1-\lambda\varphi & -U\lambda\varphi \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \Phi \\ \Psi \end{pmatrix}\right]^+ + \left[\begin{pmatrix} 1 & 0 \\ -U\lambda\varphi & 1-\lambda\varphi \end{pmatrix}\begin{pmatrix} \Phi \\ \Psi \end{pmatrix}\right]^- . \tag{4.6}$$

Relation (4.6) is extremely similar in appearance to (3.6). However, in

order to use the method described following (3.6), we have to expand (4.6) somewhat by introducing two auxiliary functions $\tilde{\Phi}$ and $\tilde{\Psi}$. Letting

$$\mathfrak{M} = \begin{pmatrix} \tilde{\Phi} & \Phi \\ \tilde{\Psi} & \Psi \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we write in place of (4.6)

$$I = \left[\begin{pmatrix} 1-\lambda\varphi & -U\lambda\varphi \\ 0 & 1 \end{pmatrix} \mathfrak{M}\right]^+ + \left[\begin{pmatrix} 1 & 0 \\ -U\lambda\varphi & 1-\lambda\varphi \end{pmatrix} \mathfrak{M}\right]^- . \tag{4.7}$$

Proceeding as before we let

$$\begin{aligned} P &= \begin{pmatrix} 1 & 0 \\ -U\lambda\varphi & 1-\lambda\varphi \end{pmatrix} \mathfrak{M} \\ Q &= \begin{pmatrix} 1-\lambda\varphi & -U\lambda\varphi \\ 0 & 1 \end{pmatrix} \mathfrak{M} . \end{aligned} \tag{4.8}$$

Then, $P - I$ has elements from A^+, Q has elements from A^-, and

$$PQ^{-1} = \frac{1}{1-\lambda\varphi} \begin{bmatrix} 1 & U\lambda\varphi \\ -U\lambda\varphi & (1-\lambda\varphi)^2 - U^2\lambda^2\varphi^2 \end{bmatrix} . \tag{4.9}$$

Thus, we are led to a matrix "factorization", Lack of commutativity in this case rules out the previous method of constructing explicit factors using exponentials. This does not necessarily mean that explicit formulas similar to (2.3) do not exist for P and Q. We will show below, however, that there do not exist formulas for P and Q with the same basic simplicity of their counterparts in (2.3). For certain special distributions of X_1, the factors P and Q can be determined. For such examples in the Bernoulli case and for the characteristic function $\varphi = 1/(1+t^2)$, the reader is referred to [8].

We return for a moment to the result of Example 2.1. Using the recurrence relation (2.6) we can write the φ_n of (2.4) as an iteration of the $+$ operator as follows:

$$\begin{aligned} \varphi_0 &= 1 \\ \varphi_1 &= \varphi^+ \\ \varphi_2 &= (\varphi\varphi^+)^+ \\ \varphi_3 &= (\varphi(\varphi\varphi^+)^+)^+ \\ &\dots \end{aligned} \tag{4.10}$$

However, formula (2.10) gives a second way of writing $\varphi_1, \varphi_2, \ldots$. That is,

$$\varphi_2 = \frac{1}{2}(\varphi^2)^+ + \frac{1}{2}\varphi^{+^2}$$

(4.11)

$$\varphi_3 = \frac{1}{3}(\varphi^3)^+ + \frac{1}{2}(\varphi^2)^+\varphi^+ + \frac{1}{6}\varphi^{+^3}$$

$$\ldots$$

Thus, we see that φ_n does not actually involve iterations of the $+$ operator but is in fact a polynomial in $\varphi^+, (\varphi^2)^+, \ldots. (\varphi^n)^+$. Moreover, these polynomials have a generating function with a closed form (2.10). The reduction of φ_n to a polynomial in $(\varphi^k)^+$ is quite clearly connected with the invariant results of type (0.2) discovered by Andersen. Let us now return to the change of sign problem.

There appear to be two possible reasons why we cannot write explicit factors P and Q of (4.9). It may be that the elements of P and Q are expressible as polynomials in $(\varphi^k)^+$, but these polynomials do not have a nice closed-form generating function. The lack of invariant results of the type (0.2) for change of sign makes this possibility unlikely. On the other hand it is possible that the elements of P and Q are not polynomials in $(\varphi^k)^+$ at all. As we will now show, this latter possibility is actually the fact.

From (4.1) we see that the coefficient of U^n in φ_n is

$$\varphi_{nn} = \int_{\substack{C_n = n \\ S_n > 0}} e^{itS_n}\, dP. \tag{4.12}$$

But, $C_n = n$ and $S_n > 0$ are possible only when n is odd. Now, if a simplification of P, or equivalently, of φ_n in (4.1) to a polynomial in $(\varphi^k)^+$ is possible, then in particular the terms

$$\varphi_{33} = (\varphi(\varphi\varphi^+)^-)^+, \quad \varphi_{55} = (\varphi(\varphi(\varphi(\varphi\varphi^+)^-)^+)^-)^+,$$

etc. will also simplify. We look first at

$$\varphi_{33} = (\varphi^2\varphi^+)^+ - (\varphi(\varphi\varphi^+)^+)^+.$$

By (4.11), $(\varphi(\varphi\varphi^+)^+)^+$ is already a polynomial in $(\varphi^k)^+$. Let us postulate the existence of constants A, B and C so that as an identity in φ

$$(4.13) \qquad (\varphi^2 \varphi^+)^+ = A(\varphi^3)^+ + B(\varphi^2)^+ \varphi^+ + C\varphi^{+3}.$$

Clearly, we need include only terms homogeneous of degree 3 on the right in (4.13). Letting $\varphi = e^{it} + 1$ in (4.13) and equating coefficients of e^{ikt} on both sides, we deduce that

$$\begin{cases} 1 = A + B + C \\ 2 = 3A + 2B \\ 1 = 3A\,. \end{cases}$$

The only solution is $A = 1/3$, $B = 1/2$ and $C = 1/6$. Thus, (4.13) becomes, with the use of (4.11),

$$(4.14) \qquad (\varphi^2 \varphi^+)^+ = \frac{1}{3}(\varphi^3)^+ + \frac{1}{2}(\varphi^2)^+ \varphi^+ + \frac{1}{6}\varphi^{+3} = (\varphi(\varphi\varphi^+)^+)^+.$$

In other words,

$$\varphi_{33} = (\varphi^2 \varphi^+)^+ - (\varphi(\varphi\varphi^+)^+)^+ = (\varphi(\varphi\varphi^+)^-)^+$$

is *identically zero* if (4.13) holds. This is clearly impossible.

In a sense we have been too restrictive in our demands for a formula for P and Q. As Andersen pointed out, the distribution of N_n depends only on the probability that S_k is positive $(k = 1, 2, \ldots, n)$. We should expect the distribution of C_n to depend at least on the probability of a change of sign at the kth step. In operator notation, this means allowing terms of the form $(\varphi(\varphi^k)_+)^-$ and $(\varphi(\varphi^k)^-)^+$ in the formula. More generally, a formula for P and Q might include products of terms of the form $(\varphi^{k_1}(\varphi^{k_2})^+(\varphi^{k_3})^+ \ldots (\varphi^{k_m})^+)^+$, which involve one iteration of the operator. Unfortunately, it can be shown by an argument similar to that used above that φ_{77} cannot be expressed as a sum of terms involving only one iteration of the operator. In general, it appears (although a proof has not been given) that φ_{nn} with $n = 2^k - 1$ cannot be reduced to an expression of terms with at most $k - 2$ iterations of the operator.

PART II

5. A difference system:

Thus far, no consideration has been given to joint distributions of the variables listed in the introduction. In this and subsequent sections we will develop a slight modification of the method of $+$ operators which is

useful for finding joint distributions. We consider here only the case in which X_k takes on only integral values, with one minor exception to this rule in Lemma 5.2. The characteristic function φ is now a Fourier series and we will use the notation

$$1 - \lambda\varphi = \sum_{k=-\infty}^{\infty} A_k e^{ikt}. \tag{5.1}$$

Once again we start with a procedure which is designed to bring the mathematical problem into sharp focus. We examine the expressions

$$\varphi_{n,k} = \int\limits_{\substack{M_k \leq n \\ L_k = 0}} e^{itS_k} dP, \qquad \left(\varphi_n = \sum_{k=0}^{\infty} \lambda^k \varphi_{n,k}\right), \tag{5.2}$$

and

$$\psi_{n,k} = \int\limits_{\substack{\overline{M}_k \geq -n \\ L_k = 0}} e^{itS_k} dP, \qquad \left(\psi_n = \sum_{k=0}^{\infty} \lambda^k \psi_{n,k}\right). \tag{5.3}$$

It is clear that $\varphi_{n,k}$ and $\psi_{n,k}$ are polynomials of at most degree n in e^{it} and e^{-it}, respectively. A fundamental property of these functions can now be stated.

Lemma 5.1: For every $n \geqq 0$,

$$\begin{aligned} \varphi_n(1-\lambda\varphi) &= \sum_{k=-\infty}^{0} C_k e^{ikt} + \sum_{k=n+1}^{\infty} C_k e^{ikt}, \qquad (C_0 = 1), \\ \psi_n(1-\lambda\varphi) &= \sum_{k=-\infty}^{-(n+1)} C'_k e^{ikt} + \sum_{k=0}^{\infty} C'_k e^{ikt}, \qquad (C'_0 = 1). \end{aligned} \tag{5.4}$$

Proof: Step I: First, we note two limit relations which were essentially proved in Part I:

$$\lim_{n\to\infty} \varphi_n = \Phi = \exp\left[\sum_{k=1}^{\infty} \frac{\lambda^k}{k} \int_{0}^{\infty} e^{itx} dP\{S_k < x\}\right], \tag{5.5}$$

and

$$\lim_{n\to\infty} \psi_n = \Psi = \exp\left[\sum_{k=1}^{\infty} \frac{\lambda^k}{k} \int_{-\infty}^{0+} e^{itx}\, dP\{S_k < x\}\right]. \tag{5.6}$$

These limits have the important product property

$$\Phi\Psi = \exp\left[\sum_{k=1}^{\infty} \frac{\lambda^k}{k} P\{S_k = 0\}\right] / [1 - \lambda\varphi]. \tag{5.7}$$

Step II: Next, we focus our attention on

$$\begin{aligned} \varphi_{n,k} - \varphi_{n-1,k} &= \int\limits_{\left\{\substack{L_k = 0 \\ M_k = n}\right\}} e^{itS_k}\, dP \\ &= \sum_{m=0}^{k} \int\limits_{\left\{\substack{L_k = 0 \\ M_k = n \\ L_k = m}\right\}} e^{itS_m} \cdot e^{it(S_k - S_m)}\, dP . \end{aligned} \tag{5.8}$$

A typical "path" satisfying the conditions of integration in the sum of (5.8) is illustrated below for $k = 7$ and $m = 3$.

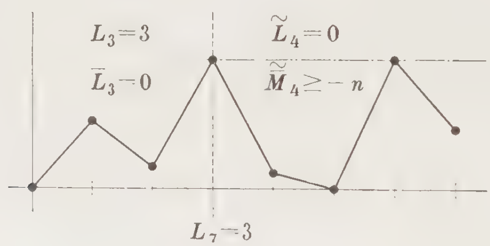


Fig. 6

Any such path can be described by separate conditions on the sets $X_1, \ldots, X_m$ and $X_{m+1}, \ldots, X_k$. Letting $\tilde{L}_{k-m}$ and $\tilde{M}_{k-m}$ denote the usual variables defined for $Y_1 = X_{m+1}, \ldots, Y_{k-m} = X_k$,

$$\left\{\begin{matrix} L_k = 0 \\ M_k = n \\ L_k = m \end{matrix}\right\} = \left\{\begin{matrix} L_m = m \\ \bar{L}_m = 0 \\ S_m = n \end{matrix}\right\} \cap \left\{\begin{matrix} \tilde{L}_{k-m} = 0 \\ \tilde{\bar{M}}_{k-m} \geq -n \end{matrix}\right\}.$$

Thus, if

(5.9) $$\alpha_{nm} = P\{\bar{L}_m = 0, \quad L_m = m, \quad S_m = n\}, \qquad \left(\alpha_n = \sum_{m=0}^{\infty} \lambda^m \alpha_{nm}\right),$$

we get

$$\varphi_{n,k} - \varphi_{n-1,k} = e^{int} \sum_{m=0}^{k} \alpha_{nm} \psi_{n,k-m},$$

or equivalently,

(5.10) $$\varphi_n - \varphi_{n-1} = \alpha_n e^{int} \psi_n .$$

By a similar argument, we find

(5.11) $$\psi_n - \psi_{n-1} = \beta_n e^{-int} \varphi_n ,$$

where

(5.12) $$\beta_{nm} = P\{L_m = 0, \ \bar{L}_m = m, \ S_m = -n\}, \qquad \left(\beta_n = \sum_{m=0}^{\infty}{}' \lambda^m \beta_{nm}\right).$$

Step III: We now observe that the known limits Φ and Ψ together with the difference equations (5.10) and (5.11) uniquely determine φ_n and ψ_n. To see this, write

(5.13) $$\begin{pmatrix} \varphi_{n-1} \\ \psi_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & -\alpha_n e^{int} \\ -\beta_n e^{-int} & 1 \end{pmatrix} \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} = \mathfrak{M}_n \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix}.$$

Without regard to justifying limits for the moment, let us iterate (5.13) to obtain

(5.14) $$\begin{pmatrix} \varphi_{n-1} \\ \psi_{n-1} \end{pmatrix} = \mathfrak{M}_n \mathfrak{M}_{n+1} \cdot \ldots \cdot \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} = \mathfrak{M} \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}.$$

It can be shown using $\Sigma |\alpha_n| < \infty$ and $\Sigma |\beta_n| < \infty$ that the matrix product in (5.14) actually exists and, moreover, the elements m_{ij} of $\mathfrak{M}$ have the form

$$m_{11} = \sum_{k=-\infty}^{0} A_k^{11} e^{ikt}, \qquad (A_0^{11} = 1),$$

(5.15) $$m_{12} = e^{int} \sum_{k=0}^{\infty} A_k^{12} e^{ikt},$$

$$m_{21} = e^{-int} \sum_{k=-\infty}^{0} A_k^{21} e^{ikt},$$

$$m_{22} = \sum_{k=0}^{\infty} A_k^{22} e^{ikt}, \qquad (A_0^{22} = 1),$$

where the series converge absolutely. Thus, from (5.14) and (5.15), using the fact that $1/\Phi$ and $1/\Psi$ are "one-sided" Fourier series

$$(5.16) \qquad \frac{\varphi_{n-1}}{\Phi\Psi} = \frac{m_{11}}{\Psi} + \frac{m_{12}}{\Phi} = \sum_{k=-\infty}^{0} B_k\, e^{ikt} + \sum_{k=n}^{\infty} B_k\, e^{ikt}.$$

A comparison of the constant terms on both sides of (5.16) together with (5.7) yields (5.4). This ends the proof of Lemma 5.1.

The basic mathematical problem of the method we will soon develop is to construct functions φ_n which have a property like (5.4) with respect to a given function. The proof above shows that the solutions φ_n and ψ_n of a difference system (5.10) and (5.11) have this rather unusual property with respect to the inverse product of their limits. A similar result is important in the continuous case (See Lemma 5.2). However, in the lattice case we can simplify the construction of φ_n and ψ_n by use of the coefficient formula for Fourier series. From (5.4)

$$(5.17) \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi_n (1 - \lambda\varphi)\, e^{imt}\, dt = \delta_{m0} \quad (m = 0, 1, \dots, n).$$

Substituting from (0.8) and (0.9) we see that in terms of the notation of (5.1) we have the explicit formulas

$$\psi_n = f_n^-(\lambda, t) \quad \text{and} \quad \varphi_n = (D_{n-1}/D_n)\, f_n^+(\lambda, t).$$

Szegö [14] has studied polynomials $g_n(z)$ and $h_n(z)$ of degree n in z and $1/z$, respectively, which are biorthogonal with respect to an integrable weight function $f(t)$ on $-\pi \leqq t \leqq \pi$. That is,

$$(5.18) \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} g_n(z)\, h_m(z)\, f(t)\, dt = \delta_{nm}, \qquad (z = e^{it}).$$

By direct substitution one can see that $e^{int} f_n^-$ and $e^{-imt} f_m^+$ of (0.8) and (0.9) have the biorthogonality property (5.18) with respect to $1 - \lambda\varphi$. For further discussion of this relation to Szegö polynomials see [7, 15, 20].

As a final consideration of this section we show how the procedure of the proof above is useful in obtaining a special but very elegant invariant result for joint distributions.

Lemma 5.2: Let $X_1, X_2, \ldots$ be independent and have a common absolutely continuous and symmetric distribution function. Then,

$$P\{N_n = n,\ \ L_n = n\} = \frac{1}{2n}. \tag{5.19}$$

Proof: Set

$$\varphi_k(x, t) = \int\limits_{\substack{M_k \leq x \\ L_k = 0}} e^{itS_k}\, dP, \qquad \left(\varphi(x, t) = \sum_{k=0}^{\infty} \lambda^k \varphi_k(x, t)\right). \tag{5.20}$$

Then, using that $P\{S_k = 0\} = 0$ $(k \geq 1)$,

$$\lim_{x\to\infty} \varphi(x, 0) = \exp\left[\sum_{k=1}^{\infty} \frac{\lambda^k}{k} (\varphi^k{}_j{}^+\right]_{t=0} = \exp\left[\sum_{k=1}^{\infty} \frac{\lambda^k}{2k}\right]. \tag{5.21}$$

Moreover, by an argument similar to (5.8) through (5.11) we have

$$\frac{d\varphi(x, t)}{dx} = \alpha(x)\, e^{itx}\, \overline{\varphi(x, t)}, \tag{5.22}$$

where bar indicates conjugate and where

$$\int_0^x \alpha(\xi)\, d\xi = \sum_{k=1}^{\infty} \lambda^k P\{L_k = k,\ \ N_k = k,\ \ S_k < x\}. \tag{5.23}$$

Letting $t = 0$ in (5.22), we get simply

$$\frac{d\varphi}{dx} = \alpha(x)\,\varphi, \qquad (\varphi(0, 0) = 1). \tag{5.24}$$

Solving (5.24) we get

$$\varphi(\infty, 0) = \exp\left[\int_0^{\infty} \alpha(\xi)\, d\xi\right] = \exp\left[\sum_{k=1}^{\infty} \frac{\lambda^k}{2k}\right] \tag{5,25}$$

and the result follows from (5.23).

6. A projection operator.

The result of Lemma 5.1 can be stated in a more convenient manner using the following notation:

Notation: For any function

$$\psi = \sum_{k=-\infty}^{\infty} C_k e^{ikt} \quad \text{with} \quad \sum |C_k| < \infty, \tag{6.1}$$

let $(m \leqq n)$

$$[\psi]_m^n = \sum_{k=m}^{n} C_k e^{ikt}. \tag{6.2}$$

In this new notation the results (5.4) become

$$[\varphi_n(1-\lambda\varphi)]_0^n = [\psi_n(1-\lambda\varphi)]_{-n}^0 = 1. \tag{6.3}$$

It is apparent that the bracket notation is actually a projection operator on the space of Fourier series (6.1). In this respect it is similar to the $+$ operator of (2.2). Unfortunately, the closure property P3. of the $+$ operator does not hold for the bracket. Nonetheless, the notation together with the next lemma will form the basis of a simple yet powerful method for working certain problems of joint distributions.

Lemma 6.1: *Let ψ be given as in (6.1) and let*

$$D_n(\psi) = \det(C_{j-i}), \qquad (i, j = 0, 1, \ldots, n).$$

If $D_n(\psi) \neq 0$, then

$$\varphi_n = \frac{1}{D_n(\psi)} \begin{vmatrix} C_0 & C_1 & \cdots & C_n \\ C_{-1} & C_0 & \cdots & C_{n-1} \\ \vdots & & \ddots & \vdots \\ C_{-n+1} & C_{-n+2} & \cdots & C_1 \\ z^n & z^{n+1} & \cdots & 1 \end{vmatrix} \qquad (z = e^{it}) \tag{6.4}$$

is a polynomial of at most degree n in e^{it} which satisfies

$$[\varphi_n \psi]_0^n = 1. \tag{6.5}$$

In the arguments below, it will be obvious that the polynomials we seek exist and are the unique solutions of equations like (6.5). Thus (6.4) will give us the explicit form of our answer. One special case of (6.4) which will be of interest later is that in which ψ is a power series, i.e. $C_k = 0$, $(k < 0)$. In this special case there exists a fixed sequence of constants $\{\alpha_k\}$ such that for every n

$$\varphi_n = \sum_{k=0}^{n} \alpha_k e^{ikt} . \tag{6.6}$$

We now turn to a few examples to illustrate the use of (6.2) and (6.4) in fluctuation problems.

Example 6.1: Maximum and all non-negative sums. Let

$$\varphi_{n,k} = \int\limits_{\left\{\begin{smallmatrix} M_k \leq n \\ L_k = 0 \end{smallmatrix}\right\}} e^{itS_k}\, dP , \qquad \left(\varphi_n = \sum_{k=0}^{\infty} \lambda^k \varphi_{n,k}\right). \tag{6.7}$$

Using φ to denote as usual the characteristic function of X_1,

$$\varphi\varphi_{n,k} = \int\limits_{\left\{\begin{smallmatrix} M_k \leq n \\ L_k = 0 \end{smallmatrix}\right\}} e^{itS_{k+1}}\, dP .$$

It follows that

$$[\varphi\varphi_{n,k}]_0^n = \int\limits_{\left\{\begin{smallmatrix} M_{k+1} \leq n \\ L_{k+1} = 0 \end{smallmatrix}\right\}} e^{itS_{k+1}}\, dP = \varphi_{n,k+1} ,$$

or equivalently,

$$\lambda [\varphi\varphi_n]_0^n = \varphi_n - 1 . \tag{6.8}$$

Now φ_n is a polynomial of at most degree n in e^{it}, and therefore

$$[\varphi_n (1 - \lambda\varphi)]_0^n = 1 . \tag{6.9}$$

By (6.4) and (0.8) the solution is

$$\varphi_n = \frac{D_{n-1}(1 - \lambda\varphi)}{D_n (1 - \lambda\varphi)} f_n^+ (\lambda , t). \tag{6.10}$$

Example 6.2: Maximum at end-point and range. Consider this time

$$\varphi_{n,k} = \int\limits_{\left\{\begin{smallmatrix} L_k = k \\ R_k \leq n \end{smallmatrix}\right\}} e^{itS_k}\, dP , \qquad \left(\varphi_n = \sum_{k=0}^{\infty} \lambda^k \varphi_{n,k}\right). \tag{6.11}$$

Let $\tilde{L}_k$ and $\tilde{R}_k$ be the usual variables defined for $X_2, \ldots, X_{n+1}$. Then

$$\varphi\varphi_{n,k} = \int_{\left\{\begin{smallmatrix}\tilde{L}_k = k \\ \tilde{R}_k \leqq n\end{smallmatrix}\right\}} e^{itS_{k+1}}\, dP\,,$$

where a step has been added at the beginning of a path. Now, $L_{k+1} = k+1$ and $R_{k+1} \leqq n$ are both satisfied for the new path if, and only if, $1 \leqq S_{k+1} \leqq n$ (See Fig. 7, below). Thus, for $n \geqq 1$ we have

Fig. 7

(6.12) $$[\varphi\varphi_{n,k}]_1^n = \varphi_{n,k+1}\,.$$

Introducing generating functions and noting that φ_n is a polynomial in e^{it} of at most degree n with constant term 1, it follows that

(6.13) $$[\varphi_n(1-\lambda\varphi)]_1^n = 0 \quad \text{and} \quad [\varphi_n]_0^0 = 1\,.$$

The solution of (6.13) is exactly $\varphi_n = f_n^+$, where f_n^+ is given in (0.8).

Example 6.3: Distribution of maximum M_n. It is interesting to see if by the present methods one can derive for lattice variables the distributions of N_n, M_n and R_{nk} found in Part I. We will illustrate how this can be done for the variable M_n. In the process we will unfold an important technique of our method. Let

(6.14) $$\varphi_{n,k} = \int_{\{M_k \leqq n\}} e^{itS_k}\, dP\,, \qquad \left(\varphi_n = \sum_{k=0}^{\infty} \lambda^k \varphi_{n,k}\right),$$

so that

$$[\varphi\varphi_{n,k}]_{-\infty}^n = \left[\int_{\{M_k \leqq n\}} e^{itS_{k+1}}\, dP\right]_{-\infty}^n = \varphi_{n,k+1}\,.$$

Thus, noting that $[\varphi_n]_{-\infty}^n = \varphi_n$, we have

$$[\varphi_n(1-\lambda\varphi)]_{-\infty}^n = 1\,. \tag{6.15}$$

We try a solution of (6.15) of the form

$$\varphi_n = \widehat{\varphi}_n \exp\Big[\sum_{k=1}^{\infty} \frac{\lambda^k}{k}(\varphi^k)^-\Big] = \widehat{\varphi}_n f_-(\lambda\,,t)\,, \tag{6.16}$$

where $\widehat{\varphi}_n$ is a polynomial in e^{it} of degree at most n. Now,

$$[\varphi_n(1-\lambda\varphi)]_{-\infty}^{-1} = \Big[\frac{\widehat{\varphi}_n}{f_+}\Big]_{-\infty}^{-1} = 0\,,$$

so that we are looking for a polynomial $\widehat{\varphi}_n$ satisfying

$$\Big[\frac{\widehat{\varphi}_n}{f_+}\Big]_0^n = 1\,. \tag{6.17}$$

Since $1/f_+$ is a power series in e^{it}, it follows from (6.6) that there is a sequence $\{\alpha_k\}$ of constants for which

$$\widehat{\varphi}_n = \sum_{k=0}^{n} \alpha_k e^{ikt}\,.$$

Moreover, since $\varphi_n \to 1/[1-\lambda\varphi]$ as n becomes infinite, by (6.16)

$$\sum_{k=0}^{\infty} \alpha_k e^{ikt} = f_+(\lambda\,,t)\,.$$

Finally, the generating function for the joint distribution of the maximum M_k and S_k is

$$\begin{aligned}\Phi &= \sum_{n=0}^{\infty} (\varphi_n - \varphi_{n-1})\,e^{in\tau}\\ &= f_-(\lambda\,,t)\sum_{n=0}^{\infty}(\widehat{\varphi}_n - \widehat{\varphi}_{n-1})\,e^{in\tau}\\ &= f_-(\lambda\,,t)\sum_{k=0}^{\infty}\alpha_k e^{ik(t+\tau)}\\ &= f_+(\lambda\,,t+\tau)\,f_-(\lambda\,,t)\,.\end{aligned} \tag{6.18}$$

Spitzer's identity (0.4) is obviously included in (6.18).

As a final remark, let us note that the bracket operator (6.2) and the method of this section has an obvious analogue for the non-lattice case. The method and arguments of the examples is exactly the same for this more general case except that no analogue of the explicit formula (6.4) seems to exist.

7. Applications of the method:

To demonstrate the power of the "projection" method of the last section, we now take up some analogues for joint distributions of the identities of Section 3.

Example 7.1: Maximum and positive partial sums. We let

$$\varphi_{n,k} = \sum_{m=0}^{k} U^m \int_{\substack{N_k = m \\ M_k \leq n}} e^{itS_k}\, dP, \qquad \left(\varphi_n = \sum_{k=0}^{\infty} \lambda^k \varphi_{n,k}\right). \tag{7.1}$$

Then, it can be shown by the usual argument that

$$U[\varphi\varphi_{n,k}]_1^n + [\varphi\varphi_{n,k}]_{-\infty}^0 = \varphi_{n,k+1}.$$

In terms of generating functions

$$[\varphi_n(1-\lambda U\varphi)]_1^n + [\varphi_n(1-\lambda\varphi)]_{-\infty}^0 = 1. \tag{7.2}$$

We try a solution of the form

$$\varphi_n = \widehat{\varphi}_n \exp\left[\sum_{k=1}^{\infty} \frac{\lambda^k}{k}(\varphi^k)^-\right], \tag{7.3}$$

where $\widehat{\varphi}$ is a polynomial of degree n in e^{it} with constant term 1. For any such polynomial $[\varphi_n(1-\lambda\varphi)]_{-\infty}^0 = 1$, so that the problem is reduced to constructing $\widehat{\varphi}_n$ satisfying

$$[\widehat{\varphi}_n \psi]_1^n = 0 \quad \text{and} \quad [\widehat{\varphi}_n]_0^0 = 1,$$

where

$$\psi = (1-\lambda U\varphi)\exp\left[\sum_{k=1}^{\infty} \frac{\lambda^k}{k}(\varphi^k)^-\right] = \sum_{k=-\infty}^{\infty} B_k e^{ikt}. \tag{7.4}$$

Thus,

$$(7.5)\qquad \widehat{\varphi}_n = \frac{1}{D_{n-1}(\psi)} \begin{vmatrix} B_0 & B_1 & \dots & B_n \\ \vdots & \ddots & & \vdots \\ B_{-n+1} & B_{-n+2} & \dots & B_1 \\ z^n & z^{n-1} & & 1 \end{vmatrix}, \qquad (z = e^{it}).$$

In summary, the generating function of the quantities in (7.1) is

$$(7.6)\qquad \varphi_n = \widehat{\varphi}_n \exp\left[\sum_{k=1}^{\infty} \frac{\lambda^k}{k} (\varphi^k)^-\right],$$

where $\widehat{\varphi}$ is given in (7.5) and (7.4).

Example 7.2: Maximum and range. To obtain information on M_n and $\overline{M}_n$ it is easiest to find the joint distribution of M_n and R_n. Clearly, the joint distribution of M_n and $\overline{M}_n$ will follow. In this case we let

$$(7.7)\qquad \varphi_{n,k} = \sum_{m=0}^{k} U^m \int_{\left\{\begin{smallmatrix} L_k = m \\ R_k \leq n \end{smallmatrix}\right\}} e^{itM_k}\, dP, \qquad \left(\varphi_n = \sum_{k=0}^{\infty} \lambda^k \varphi_{n,k}\right).$$

By the bracket method, adding a step to the beginning of a path, it can be shown that

$$(7.8)\qquad \begin{aligned} U[\varphi\varphi_{n,k}]_1^n &= \sum_{m=1}^{k+1} U^m \int_{\left\{\begin{smallmatrix} L_{k+1} = m \\ R_{k+1} \leq n \end{smallmatrix}\right\}} e^{itM_{k+1}}\, dP \\ &= \varphi_{n,k+1} - P\{L_{k+1} = 0,\ \ R_{k+1} \leq n\}. \end{aligned}$$

The generating function $\psi_n(0)$ of the second term on the right in (7.8) was evaluated in Section 5. We found $\psi_n(0) = f_n^-(\lambda, 0)$, where $f_n^-(\lambda, t)$ is given in (0.9). Thus, from (7.8)

$$\varphi_n = \psi_n(0) + \lambda U[\varphi\varphi_n]_1^n,$$

or equivalently,

$$(7.9)\qquad [\varphi_n(1 - \lambda U\varphi)]_1^n = 0 \quad \text{and} \quad [\varphi_n]_0^0 = \psi_n(0).$$

The solution of (7.9) is $\varphi_n = \psi_n(0)\, f_n^+(\lambda U, t)$, where $f_n^+(\lambda, t)$ is given by (0.8). In summary, the generating function of the quantities defined by (7.7) is

$$\varphi_n = f_n^+(\lambda U, t)\, f_n^-(\lambda, 0). \tag{7.10}$$

Example 7.3: First positive sum with restricted minimum. For our final example we consider the evaluation of

$$\psi_{n,k} = \int_{\left\{\begin{matrix} N_{k-1}=0 \\ \overline{M}_k \geq -n \\ S_k > 0 \end{matrix}\right\}} e^{itS_k}\, dP, \qquad \left(\psi_n = \sum_{k=1}^{\infty} \lambda^k \psi_{n,k}\right). \tag{7.11}$$

But in Section 5, we showed that

$$\varphi_{n,k} = \int_{\left\{\begin{matrix} N_k = 0 \\ M_k \geq -n \end{matrix}\right\}} e^{itS_k}\, dP$$

has generating function $\varphi_n = f_n^-(\lambda, t)$. Thus

$$[\varphi\varphi_{n,k}]_{-n}^{\infty} = \varphi_{n,k+1} + \psi_{n,k+1}, \tag{7.12}$$

or equivalently,

$$\psi_n = 1 - [\varphi_n(1-\lambda\varphi)]_{-n}^{\infty}. \tag{7.13}$$

Now $[\psi_n]_1^{\infty} = \psi_n$, so that actually

$$[\varphi_n(1-\lambda\varphi)]_{-n}^{0} = 1 \tag{7.14}$$

and

$$\psi_n = 1 - [(1-\lambda\varphi)\varphi_n]_0^{\infty}. \tag{7.15}$$

Relation (7.14) is the one which determines that $\varphi_n = f_n^-(\lambda, t)$. In summary, the generating function of the quantities defined in (7.11) is

$$\psi_n = 1 - [(1-\lambda\varphi)\, f_n^-(\lambda, t)]_0^{\infty}. \tag{7.16}$$

8. Examples:

To illustrate the results previously derived we consider here a series of three specific examples. The computations involved are straightforward applications of the method described in [14, § 5.3] and will be omitted. We will use the notation

$$(8.1) \qquad \tilde{f}_n^+(z) = \begin{vmatrix} A_0 & A_1 & \cdots & A_{n-1} & A_n \\ A_{-1} & A_0 & \cdots & A_{n-2} & A_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \\ A_{-n+1} & A_{-n+2} & \cdots & A_0 & A_1 \\ z^n & z^{n-1} & \cdots & z & 1 \end{vmatrix}$$

(with a similar formula for $\tilde{f}_n^-(z)$) to denote the numerator of the expression $f_n^+(z)$ given in (0.8). Note that $D_n(1-\lambda\varphi) = \tilde{f}_{n+1}^+(0)$ and that

$$f_n^+(z) = \tilde{f}_n^+(z) / \tilde{f}_n^+(0).$$

Example 1: Bernoulli variables. Let X_k have the distribution $P\{X_k = 1\} = p$ and $P\{X_k = -1\} = q$. The determinant in (8.1) becomes essentially bordered. We find that

$$(8.2) \qquad \begin{aligned} \tilde{f}_n^+(z) &= \sum_{k=0}^{n} \lambda^k p^k z^k (r_1^{n+1-k} - r_2^{n+1-k}) / (r_1 - r_2) \\ &= \left[\frac{r_1^{n+1} - (\lambda p z)^{n+1}}{1 - \dfrac{\lambda p z}{r_1}} - \frac{r_2^{n+1} - (\lambda p z)^{n+1}}{1 - \dfrac{\lambda p z}{r_2}} \right] \frac{1}{r_1 - r_2}, \end{aligned}$$

where

$$r_1 = \frac{1 + \sqrt{1 - 4\lambda^2 pq}}{2}, \quad r_2 = \frac{1 - \sqrt{1 - 4\lambda^2 pq}}{2}.$$

The formula for $\tilde{f}_n^-(z)$ is found from (8.2) by interchanging p and q and replacing z by $1/z$.

Let us examine the results of Section 5 in terms of the formula (8.2). From (5.2), (6.10) and (8.2) we see that ($z = e^{it}$)

$$(8.3) \qquad \begin{aligned} \varphi_n &= \sum_{k=0}^{\infty} \lambda^k \int_{\left\{ \substack{M_k \leq n \\ L_k = 0} \right\}} e^{itS_k} \, dP \\ &= \left[\frac{r_1^{n+1} - (\lambda p z)^{n+1}}{1 - \dfrac{\lambda p z}{r_1}} - \frac{r_2^{n+1} - (\lambda p z)^{n+1}}{1 - \dfrac{\lambda p z}{r_2}} \right] / (r_1^{n+2} - r_2^{n+2}). \end{aligned}$$

Moreover, system (5.10—5.11) will be satisfied by φ_n and the corresponding ψ_n where α_n and β_n are given by

$$\alpha_n = \frac{\lambda^n p^n (r_1 - r_2)}{r_1^{n+1} - r_2^{n+1}}, \quad \beta_n = \frac{\lambda^n q^n (r_1 - r_2)}{r_1^{n+1} - r_2^{n+1}}. \tag{8.4}$$

Finally, let us observe that $f_+(\lambda, t)$ and $f_-(\lambda, t)$ of (0.6—0.7) can be found from (8.2) by letting n become infinite. We have

$$\begin{aligned} f_+ &= \lim_{n\to\infty} \frac{\tilde{f}_n^+(z)}{\tilde{f}_n^+(0)} = \frac{1}{1 - \dfrac{\lambda p}{r_1} e^{it}} \\ f_- &= \lim_{n\to\infty} \frac{\tilde{f}_n^-(z)}{\tilde{f}_{n+1}^-(0)} = \frac{1}{r_1 - \lambda q e^{-it}}. \end{aligned} \tag{8.5}$$

It follows that $f_+ f_- = 1 / (1 - \lambda p e^{it} - \lambda q e^{-it})$ and, of course, the conditions of Lemma 1.1 are satisfied. Inversions of the transforms seem particularly easy in this case.

Example 2: Two-sided geometric. For our next example we take the distribution

$$\begin{cases} P\{X_k = \pm m\} = \dfrac{1}{2} p q^m & (m \geqq 1) \\ P\{X_k = 0\} = p. \end{cases} \tag{8.6}$$

It turns out that $\tilde{f}_n(z)$ in this case has the form

$$\begin{aligned} \tilde{f}_n^+(z) = (zq)^{n+1}\left(1 - \lambda\frac{p}{2}\right)^n &+ (1 - zq)\frac{r_1 - q^2}{r_1 - r_2}\sum_{k=0}^{n}\left[zq\left(1 - \lambda\frac{p}{2}\right)\right]^k r_1^{n-k} \\ &- (1 - zq)\frac{r_2 - q^2}{r_1 - r_2}\sum_{k=0}^{n}\left[zq\left(1 - \lambda\frac{p}{2}\right)\right]^k r_2^{n-k}, \end{aligned} \tag{8.7}$$

where

$$r_1 = \frac{(1 - \lambda p + q^2) + \sqrt{(1 - \lambda p + q^2)^2 - 4q^2\left(1 - \lambda\dfrac{p}{2}\right)^2}}{2}$$

$$r_2 = \frac{(1 - \lambda p + q^2) - \sqrt{(1 - \lambda p + q^2)^2 - 4q^2\left(1 - \lambda\dfrac{p}{2}\right)^2}}{2}.$$

By appropriate division, the various quantities $f_n^+(z)$, $f_n^-(z)$, etc. can be

constructed from (8.7). We find $\tilde{f}_n^-(z)$ from (8.7) simply by replacing z by $1/z$. As n becomes infinite

(8.8)
$$f_n^+(z) \to f_+ = \frac{1 - zq}{1 - \dfrac{zq}{r_1}\left(1 - \lambda\dfrac{p}{2}\right)}, \qquad (z = e^{it}),$$

$$f_n^-(z) \to f_- = \frac{1 - \dfrac{q}{z}}{r_1 - \dfrac{q}{z}\left(1-\lambda\dfrac{p}{2}\right)}, \qquad (z = e^{it}).$$

Example 3: Two-sided exponential. Finally, we consider a sequence $\{X_k\}$ whose distributions are continuous, each having density $f(x) = \frac{1}{2}e^{-|x|}$ $(-\infty < x < \infty)$. The computations in this case can be deduced from Example 2 by an appropriate limiting process. We take $q = 1-p = e^{-\Delta x}$ $z = e^{i\xi\Delta x} = e^{it}$, $n = x/\Delta x$ and $k = y/\Delta x$, and then let $\Delta x \to 0$. In this way we pass from Fourier series in e^{it} in Example 2 to Fourier transforms in $e^{i\xi x}$ in this example. The counterpart of (8.1) exists under this limit and we find that

(8.9)
$$\begin{aligned}\tilde{f}_+(x,\xi) = {} & \frac{\sigma^2}{4\sqrt{1-\lambda}}\, e^{-\left(\frac{\lambda}{2}+\mu\right)x} - \frac{\mu^2}{4\sqrt{1-\lambda}}\, e^{-\left(\frac{\lambda}{2}+\sigma\right)x} \\ & + \frac{\sigma\lambda}{4\sqrt{1-\lambda}}\, e^{-\left(\frac{\lambda}{2}+\mu\right)x} \cdot \frac{e^{(i\xi-\sqrt{1-\lambda})x} - 1}{i\xi - \sqrt{1-\lambda}} \\ & - \frac{\mu\lambda}{4\sqrt{1-\lambda}}\, e^{-\left(\frac{\lambda}{2}+\sigma\right)x} \cdot \frac{e^{(i\xi+\sqrt{1-\lambda})x} - 1}{i\xi + \sqrt{1-\lambda}},\end{aligned}$$

where

$$\sigma = 1 + \sqrt{1-\lambda}, \quad \mu = 1 - \sqrt{1-\lambda}.$$

The counterpart of the determinant $D_n(1-\lambda\varphi) = \tilde{f}_{n+1}^+(0)$ is found by letting $i\xi \to -\infty$. We thus find

(8.10)
$$D(x, 1-\lambda\varphi) = \frac{\sigma^2}{4\sqrt{1-\lambda}}\, e^{-\left(\frac{\lambda}{2}+\mu\right)x} - \frac{\mu^2}{4\sqrt{1-\lambda}}\, e^{-\left(\frac{\lambda}{2}+\sigma\right)x}.$$

Dividing (8.9) by (8.10) gives a number of quantities with very interesting probability interpretation. By formal analogy with the result

$$f_n^+(z) = \tilde{f}_n^+(z) / \tilde{f}_n^+(0)$$

and its use in Examples 6.1, 7.2 and (5.2), we deduce that for the variables X_k of this example

$$\begin{aligned}\varphi(x, \xi) &\equiv \sum_{k=0}^{\infty} \lambda^k \int\limits_{\substack{M_k \leqq x \\ \overline{M}_k = 0}} e^{i\xi S_k}\, dP \\ &= \frac{\tilde{f}_+(x, \xi)}{D(x, 1-\lambda\varphi)} \\ &= \overline{\sum_{k=0}^{\infty} \lambda^k \int\limits_{\substack{M_k = 0 \\ \overline{M}_k \geqq -x}} e^{i\xi S_k}\, dP} = \overline{\psi(x, \xi)}.\end{aligned} \tag{8.11}$$

We expect that the $\varphi(x, \xi)$ and $\psi(x, \xi)$ of (8.11) satisfy a system analogous to (5.10—5.11). A tedious calculation will show that for

$$\alpha(x) = \frac{\lambda e^{-\left(\frac{\lambda}{2}+1\right)x}}{2D(x, 1-\lambda\varphi)}, \tag{8.12}$$

the function $\varphi(x, \xi)$ satisfies

$$\frac{d\varphi(x, \xi)}{dx} = \alpha(x)\, e^{i\xi x}\, \overline{\varphi(x, \xi)}. \tag{8.13}$$

Taking limits as x becomes infinite, we find, moreover,

$$\begin{aligned}f_+(\xi) &= \lim_{x \to \infty} \varphi(x, \xi) = 1 - \frac{\lambda}{1+\sqrt{1-\lambda}} \cdot \frac{1}{i\xi - \sqrt{1-\lambda}} \\ &= \frac{1}{1 - \dfrac{1+\sqrt{1-\lambda}}{1-i\xi}}.\end{aligned}$$

Clearly, $|f_+(\xi)|^2 = [1 - \lambda/(1+\xi^2)]^{-1}$, as was expected.

REFERENCES

[1] E. Sparre Andersen, "On the number of positive sums of random variables", *Skand. Aktuarietid*, 32, pp. 27—36 (1949).

[2] E. Sparre Andersen, "On the frequency of positive partial sums of a series of random variables", *Mathematisk Tidsskrift*, pp. 33—35 (1950).

[3] E. Sparre Andersen, "On sums of symmetrically dependent random variables", *Skand. Aktuarietid.* 36, pp. 123—138 (1953).

[4] E. Sparre Andersen, "On the fluctuations of sums of random variables", *Math. Scand.* 1, pp. 263—285 (1953).

[5] E. Sparre Andersen, "On the fluctuations of sums of random variables II", *Math. Scand.* 2, pp. 195—223 (1954).

[6] Glen Baxter, "An operator identity", *Pacif. Jour. Math.* 4, pp. 649—663 (1958).

[7] Glen Baxter, "Polynomials defined by a difference system". *Bulletin Amer. Math. Soc.* 66, pp. 187—190 (1960).

[8] Glen Baxter, "A two-dimensional operator identity with application to the change of sign in sums of random variables", *Trans. Amer. Math. Soc.* 96, pp. 210—221 (1960).

[9] K. L. Chung and M. Kac, "Remarks on fluctuations of sums of independent random variables", *Mem. Amer. Math. Soc.*, No. 6, pp. 1—11 (1950).

[10] D. A. Darling, "Sums of symmetrical random variables", *Proc. Amer. Math. Soc.* 2, pp. 511—517 (1951).

[11] P. Erdös and M. Kac, "On the number of positive sums of independent random variables", *Bull. Amer. Math., Soc.* 53, pp. 1011—1020 (1947).

[12] W. Feller, "The number of zeros and changes of sign in a symmetric random walk", *L'Enseignement Mathématique*, tome III, fasc. 3, pp. 229—235 (1957).

[13] W. Feller, "On combinatorial methods in fluctuation theory", Harold Cramér Volume, pp. 75—91 (1959).

[14] U. Grenander and G. Szegö, "Toeplitz torms and their applications", Univ. Calif. Press, Berkeley 1958.

[15] M. Kac, "Toeplitz matrices, translation kernels, and a related problem in probability theory", *Duke Math. Jour.* 21, pp. 501—509 (1954).

[16] F. Pollaczek, "Fonctions caracteristiques de certaines répartitions définies au moyen de la notion d'ordre. Application à la théorie des attentes", *C. R. Acad. Sci. Paris* 234, pp. 2334—2336 (1952).

[17] D. Ray, "Stable processes with an absorbing barrier", *Trans. Amer. Math. Soc.* 89, pp. 16—24 (1958).

[18] F. Spitzer, "A combinatorial lemma and its application to probability theory", *Trans. Amer. Math. Soc.* 82, pp. 323—339 (1956).

[19] F, Spitzer, "The Wiener-Hopf equation whose kernel is a probability density", *Duke. Math. Jour.* 24, pp. 327—343 (1957).

[20] F. Spitzer and C. Stone, "A class of Toeplitz forms and their application to probability theory", *Ill. Jour. Math.* 4, pp. 253—277 (1960).

[21] J. G. Wendel, "Spitzer's formula: a short proof", *Proc. Amer. Math. Soc.* 9, pp. 905—908 (1958).

[22] J. G. Wendel, "Order statistics of partial sums", *Annals. Math. Stat.* 31, pp. 1034—1044 (1960).

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(Received October 8, 1960)

ON THE BOREL EXCEPTIONAL VALUES OF LACUNARY INTEGRAL FUNCTIONS

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1. In his paper: [1] "Ueber die Wurzel vom kleinsten absoluten Betrage einer algebraischen Gleichung" Fejer proved in 1908 that a lacunary integral function

$$f(z) = \sum_{1}^{\infty} a_n z^{\lambda_n}$$

which satisfies the gap-condition:

$$\sum_{1}^{\infty} \frac{1}{\lambda_n} < \infty \tag{1.1}$$

must have a zero. (This implies of course that $f(z)$ takes every complex value.)

The question naturally arises to what extent can this interesting result be extended or improved. Biernacki [2] proved in 1927 that an integral function subject to the gap-condition (1.1) can have no 'Picard exceptional values', i.e. $f(z)$ takes every complex value infinitely often. On the other hand for functions of finite order much stronger results can be obtained.

Pólya [3] pointed out in 1929 that for such functions the gap-condition:

$$\limsup (\lambda_{n+1} - \lambda_n) = \infty \tag{1.2}$$

is already incompatible with the existence of a Picard exceptional value. This result was substantially strengthened in 1935 by Pfluger and Pólya himself who proved [4], that (again for functions of finite order) the gap-condition:

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1. Fejer [4].
2. Biernacki [1].
3. Pólya [6].
4. Pfluger and Pólya [13].

$$\limsup \frac{\lambda_n}{n} = \infty \tag{1.3}$$

which is only slightly stronger than (1.2) is incompatible even with the existence of a Borel exceptional value.

The main purpose of this paper is to prove that if either of the gap-conditions : [5]

$$\frac{\lambda_{n+1} - \lambda_n}{\varphi(\log \lambda_n)} \to \infty \tag{1.4}$$

$$\frac{\lambda_n}{\varphi(n) \log \log n} \to \infty , \tag{1.5}$$

where $\varphi(x)$ is virtually any function subject to the condition :

$$\int_0^\infty \varphi(x)^{-1}\, dx < \infty ,$$

is satisfied, $f(z)$ can have no Borel exceptional value. Both these conditions imply the condition (1.1). However it will be seen, that if the growth of $f(z)$ is restricted, (1.4) and (1.5) can be replaced respectively by weaker conditions, which do not imply the condition (1.1).

2. The main results, referred to in the introduction, are contained in the following theorems :

Theorem 1. *Let*

$$f(z) = \sum_1^\infty a_n z^{\lambda_n}$$

be an integral function of infinite order, which has $z = 0$ as a Borel exceptional value. (By this we mean that the sequence of its zeros has a finite exponent of convergence.)

Then we have that:

5. Note that the gap-conditions (1.4) and (1.5) are independent (neither implies the other), though, in a sense (1.4) is the weaker (i.e. better) one. It is unfortunately inherent in the method used, that (1.5) can not be replaced by the sharp condition : $\dfrac{\lambda_n}{\varphi(n)} \to \infty$ which one would expect.

$$\varlimsup_{n\to\infty} \frac{\lambda_{n+1} - \lambda_n}{(\log \lambda_n)\,\Omega(\log \lambda_n)} < \infty \tag{2.1}$$

for any positive, increasing function $\Omega(x)$, defined for $x \geqq 0$, and satisfying the conditions:

$$\int_0^\infty \frac{dx}{x\Omega(x)} < \infty \tag{2.2}$$

$$\Omega(x^\lambda) \leq \lambda^K \Omega(x) \tag{2.3}$$

for $x > 0$, $\lambda > 1$, with a suitable constant K.

Theorem 2. Under the same conditions as in Theorem 1:

$$\varliminf_{n\to\infty} \frac{\lambda_n}{n \cdot \Omega(n) \cdot \log\log n} < \infty\,. \tag{2.4}$$

Theorem 3.[6] If in addition to the assumption made in Theorem 1, the growth of: $M(r) = \max\limits_{|z|=r} f(z)$ is restricted by the condition:

$$\varlimsup_{r\to\infty} \frac{\log\log\log M(r)}{\log r} = \rho < \infty \tag{2.5}$$

then, (2.1) can be replaced by:

$$\varliminf_{n\to\infty} \frac{\lambda_{n+1} - \lambda_n}{\log \lambda_n} < \infty\,. \tag{2.6}$$

Theorem 4.[6] If in addition to the assumption made in Theorem 1, $M(r)$ is subject to condition (2.5), then, (2.4) can be replaced by:

$$\varliminf_{n\to\infty} \frac{\lambda_n}{n \cdot \log n \cdot \log\log n} < \infty\,. \tag{2.7}$$

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6. An analysis of the proof makes it clear that while any restriction on $M(r)$ leads to a significant improvement of (2.1), a restriction weaker than (2.5), does not lead to an essential improvement of (2.4). This is becaure of the presence of the factor $\log\log n$ in (2.4).

3. We state now three results, due respectively to Wiener, Turán, and Wiman-Valiron, which form the starting point of our proofs.

Theorem A (Wiener's lemma):[7] If $\{\lambda_n\}$ is a strictly increasing sequence of non-negative integers, and:

$$\lambda_{n+1} - \lambda_n \geqq L \qquad n = 1, 2, \ldots \tag{3.1}$$

$$\delta = \frac{16\pi}{L}$$

and the Fourier series (with complex coefficients):

$$\varphi(t) = \sum_1^\infty a_n e^{i\lambda_n t}$$

converges absolutely, then for every real ϑ

$$\int_{\Theta-(\delta/2)}^{\Theta+(\delta/2)} |\varphi(t)|^2 \, dt \geqq \frac{1}{2L} \cdot \int_0^{2\pi} |\varphi(t)|^2 \, dt . \tag{3.2}$$

Theorem B (Turán's lemma):[8] If $\{\lambda_n\}$ is a strictly increasing sequence of non-negative integers, then for every

$$0 \leqq \vartheta < 2\pi, \quad 0 < \delta \leqq 2\pi$$

$$\max_{|z|=r} \left| \sum_{n=1}^M a_n z^{\lambda_n} \right| \leqq \left(\frac{48\pi}{\delta} \right)^M \max_{\substack{|z|=r \\ \vartheta-\delta/2 \leqq \arg z \leqq \vartheta+\delta/2}} \left| \sum_{n=1}^M a_n z^{\lambda_n} \right| . \tag{3.3}$$

Theorem C (The Wiman-Valiron lemma):[9] Let $\{R_n\}$ be a strictly increasing sequence of positive numbers, and let: $R_n \to \bar{R}$ as $n \to \infty$ ($\bar{R}$ may be infinite). The power series:

$$F(Z) = \sum_0^\infty A_n Z^n = \sum_0^\infty \frac{Z^n}{R_1 R_2 \ldots R_n} \qquad (A_0 = 1) \tag{3.4}$$

will converge for $|Z| = R < \bar{R}$.

7. Wiener [10].

8. Turán [8] p. 30. Turán [8] gives numerous applications of this very interesting lemma. The constant 48 can be replaced by $4e$.

9. Saxer [7]. The result here is stated only for $\bar{R} < \infty$, but the proof is valid also for $\bar{R} = \infty$. To the best of the author's knowledge, the case $\bar{R} = \infty$ was first used by J. Clunie ([2]).

Let

$$(3.5) \qquad f(z) = \sum_0^\infty c_n z^n$$

and

$$(3.6) \qquad \varphi(\zeta) = \sum_0^\infty \frac{c_n}{A_n} \zeta^n = \sum_0^\infty c_n R_1 R_2 \dots R_n \zeta^n$$

be integral functions. (If $R < \infty$, then the assumption that $f(z)$ is an integral function, will automatically ensure that $\varphi(\zeta)$ is also an integral function.) Let $N(r, f)$, $N(R, F)$, $N(\rho, \varphi)$, $\mu(r, f)$, $\mu(R, F)$, and $\mu(\rho, \varphi)$ denote the central-index and maximal-term of the functions: $f(z)$, $F(Z)$, $\varphi(\zeta)$ respectively.

Under these conditions there exists a sequence of non-negative integers:

$$n_1 < n_2 < n_3 < \dots < n_k < \dots$$

and a sequence of positive numbers:

$$0 = r_{n_1} < r'_{n_1} < r_{n_2} < r'_{n_2} < \dots < r_{n_k} < r'_{n_k} < \dots$$

such that:

$$(3.7) \qquad \sum_{k=1}^{m-1} \log \frac{r_{n_{k+1}}}{r'_{n_k}} = \log R_{n_m} - \log R_{n_1}$$

and with the property, that for every r in the interval $r_{n_\nu} < r < r'_{n_\nu}$:

$$(3.8) \qquad \text{(i)} \quad \begin{aligned} \frac{|c_{n-l}|\, r^{n-l}}{|c_n|\, r^n} &= \frac{R_{n-l+1} \dots R_{n-1} R_n}{R_n^l} = \frac{A_{n-l} R_n^{n-l}}{A_n R_n^n} \\ \frac{|c_{n+k}|\, r^{n+k}}{|c_n|\, r^n} &< \frac{R_n^k}{R_{n+1} R_{n+2} \dots R_{n+k}} = \frac{A_{n+k} R_n^{n+k}}{A_n R_n^n} \end{aligned}$$

for every: $l > 0$, $k > 0$, $n = n_\nu$

$$(3.9) \qquad \text{(ii)} \quad N(r, f) = N(R_{n_\nu}, F) = N\left(\frac{r}{R_{n_\nu}}, \varphi\right) = n_\nu$$

$$(3.10) \qquad \text{(iii)} \quad \mu(r, f) = \mu(R_{n_\nu}, F) \cdot \mu\left(\frac{r}{R_{n_\nu}}, \varphi\right).$$

The integers: $\{n_k\}$ are referred to as admissible integers, the values r, lying inside one of the admissible intervals: $r_{n_\nu} < r < r'_{n_\nu}$ $(\nu = 1, 2, \ldots)$ are referred to as ordinary values, while the complementary intervals, and the values contained in them are the exceptional intervals, and exceptional values respectively.

Corollary I: For every ordinary r, such that: $r \geqq r^*$, where r^* depends only on $f(z)$, we have

$$\mu(r, f) \geqq \mu(R_N, F)$$

where $N = N(r, f)$.

In fact $\lim\limits_{\rho\to\infty} \mu(\rho, \varphi) = \infty$, hence

$$\mu\left(\frac{r}{R_N}, \varphi\right) \geqq 1, \qquad \text{if} \quad \frac{r}{R_N} \geqq \rho^*$$

$$\frac{r}{R_N} \geqq \rho^*, \qquad \text{if} \quad N\left(\frac{r}{R_N}, \varphi\right) = N \geqq N^*$$

$$N = N(r, f) \geqq N^*, \qquad \text{if} \quad r \geqq r^*.$$

Summing up:

$$\mu\left(\frac{r}{R_N}, \varphi\right) = \frac{\mu(r, f)}{\mu(R_N, F)} \geqq 1, \qquad \text{if} \quad r \geqq r^*.$$

Corollary II. If $\overline{R} < \infty$, the set of exceptional values is of (finite) logarithmic measure $\log \overline{R} - \log R_{n_1}$.

4. We shall also need a number of additional lemmas (derived partly from the above key-lemmas). Some of these lemmas — notably Lemma VIII, and Part (i) of Lemma II — are of independent, intrinsic interest.

Lemma I. Let $\psi(x)$ be any positive increasing function, such that

$$\psi(\lambda x) \leqq \lambda^K \psi(x) \tag{4.1}$$

for every $x > 0$, $\lambda \geqq 1$ (with a suitable K, depending only on the function). We write

$$\alpha(x) = -\beta(x) = \int\limits_{\log x}^{\infty} \frac{dt}{\psi(t)} \tag{4.2}$$

if

$$\int_0^\infty \frac{dt}{\phi(t)} < \infty ;$$

(4.3) $$\beta(x) = -\alpha(x) = \int_1^{\log x} \frac{dt}{\phi(t)},$$

if

$$\int_0^\infty \frac{dt}{\phi(t)} = \infty$$

(4.4) and $$R_n = e^{\beta(n)}.$$

We have then for $n > n_0$ the following inequalities

(4.5) $$\log \frac{R_n^n}{R_1 R_2 \dots R_n} > \frac{n}{3\phi(\log n)}$$

$$\sum_{l=m}^{n-1} \frac{R_{n-l} \dots R_{n-1} R_n}{R_n^{l+1}} <$$

(4.6) $$< \frac{n}{m-1} \phi(\log n) \exp\left\{-\frac{(m-1)^2}{2n\phi(\log n)}\right\}$$

$$\sum_{k=m+1}^{\infty} \frac{R_n^k}{R_{n+1} R_{n+2} \dots R_{n+k}} <$$

(4.7) $$< m\left(\frac{n+m}{m}\right)^4 \exp\left\{-\frac{m^2}{2(n+m)\phi(\log(n+m))}\right\}.$$

In fact, (4.5) and (4.6) (but not (4.7)) are valid without the assumption (4.1).

Proof of the lemma:

(4.8) $$x\beta'(x) = \frac{1}{\phi(\log x)}.$$

Hence $x\beta'(x)$ and $\beta'(x)$ are positive decreasing functions, while $\beta(x)$ increases, and $\alpha(x)$ decreases.

$$\lim_{x\to\infty} \beta(x) = 0 \qquad \text{if} \quad \int_0^\infty \frac{dt}{\psi(t)} < \infty \tag{4.9}$$

$$\lim_{x\to\infty} \beta(x) = +\infty \qquad \text{if} \quad \int_0^\infty \frac{dt}{\psi(t)} = \infty\,. \tag{4.10}$$

Let us write

$$h(x) = x\,\beta(n) - \int_0^x \beta(n+t)\,dt = \int_0^x \{\beta(n) - \beta(n+t)\}\,dt$$

$$h'(x) = \beta(n) - \beta(n+x)$$

$$h''(x) = -\beta'(n+x)\,.$$

By Taylor's formula: for $x > 0$

$$h(x) = h(0) + h'(0)\,x + \frac{1}{2} h''(\xi)\,x^2 = \frac{1}{2} h''(\xi)\,x^2 = -\frac{1}{2}\beta'(n+\xi)\,x^2$$

with $0 < \xi < x$

$$h(-x) = \frac{1}{2} h''(-\xi')\,x^2 = -\frac{1}{2}\beta'(n-\xi')\,x^2$$

with $0 < \xi' < x$.

Since $\beta'(x)$ is decreasing, we find that, for $x > 0$

$$h(x) < -\frac{1}{2}\beta'(n+x)\,x^2 \tag{4.11}$$

$$h(-x) < -\frac{1}{2}\beta'(n)\,x^2\,. \tag{4.12}$$

Since $\beta(x)$ is increasing, we obtain now, using (4.12), that

$$\log \frac{R_{n-l} \ldots R_{n-1} R_n}{R_n^{l+1}} = \log \frac{R_{n-l} \ldots R_{n-1}}{R_n^l} \tag{4.13}$$

$$= \beta(n-l) + \ldots + \beta(n-2) + \beta(n-1) - l\,\beta(n)$$

$$< \int_{n-l}^{n} \beta(x)\,dx - l\beta(n) = -l\beta(n) - \int_0^{-l} \beta(n+t)\,dt = h(-l)$$

$$< -\frac{1}{2}\beta'(n)\,l^2 = -\frac{l^2}{2n\psi(\log n)}\,.$$

Similarly, from (4.11)

$$\log \frac{R_n^k}{R_{n+1} R_{n+2} \dots R_{n+k}} \tag{4.14}$$

$$= k\beta(n) - \{\beta(n+1) + \beta(n+2) + \dots + \beta(n+k)\}$$

$$< k\beta(n) - \int_n^{n+k} \beta(x)\,dx = k\beta(n) - \int_0^k \beta(n+t)\,dt = h(k)$$

$$< -\frac{1}{2}\beta'(n+k)\,k^2 = -\frac{k^2}{2(n+k)\,\phi(\log(n+k))}$$

Substituting $l = n-1$ into (4.13)

$$\log \frac{R_1 R_2 \dots R_n}{R_n^n} < -\frac{(n-1)^2}{2n\phi(\log n)} < -\frac{n}{3\phi(\log n)}$$

for sufficiently big n, which proves (4.5).

From (4.13) we also obtain that

$$\sum\nolimits_1 = \sum_{l=m}^{n-1} \frac{R_{n-l} \dots R_n}{R_n^{l+1}} < \sum_{l=m}^{n-1} \exp\left\{-\frac{l^2}{2n\phi(\log n)}\right\}$$

$$< \int_{m-1}^{\infty} \exp\left\{-\frac{x^2}{2n\phi(\log n)}\right\} dx.$$

Substituting

$$u = \frac{x^2}{2n\phi(\log n)}, \quad dx = n\phi(\log n)\frac{du}{x}$$

we obtain that

$$\sum\nolimits_1 < \int_{\frac{(m-1)^2}{2n\phi(\log n)}}^{\infty} e^{-u}\, n\phi(\log n)\frac{du}{x} \leqq \frac{n\phi(\log n)}{m-1}\, e^{-\frac{(m-1)^2}{2n\phi(\log n)}}$$

which proves (4.6).

Further, if we write:

$$\lambda(x) = \frac{x}{\phi(\log x)}; \quad u = u(x) = \frac{1}{2}\left(\frac{x}{n+x}\right)^2 \lambda(n+x)$$

we find, by virtue of (4.1), that, for $\mu > 1$

$$\lambda(x^\mu) = \frac{x^\mu}{\psi(\mu \log x)} \geqq \frac{x^\mu}{\mu^K \psi(\log x)} \geqq \frac{x(x \cdot e^{-K})^{\mu-1}}{\psi(\log x)}$$

$$= \lambda(x)(x \cdot e^{-K})^{\mu-1}$$

$$\frac{\lambda(x^\mu)-\lambda(x)}{x^\mu - x} \geqq \frac{\lambda(x)}{x} \frac{(x \cdot e^{-K})^{\mu-1}-1}{x^{\mu-1}-1}$$

$$\lambda'(x) \geqq \frac{1}{\psi(\log x)} \lim_{\mu\to 1} \frac{(x \cdot e^{-K})^{\mu-1}-1}{x^{\mu-1}-1}$$

$$= \frac{1}{\psi(\log x)} \frac{\log x - K}{\log x} > \frac{1}{2\psi(\log x)} > 0$$

if $x > e^{2K}$

$$\frac{du}{dx} = \frac{1}{2}\left(\frac{x}{n+x}\right)^2 \lambda'(n+x) + \frac{xn}{(n+x)^3}\lambda(n+x) \tag{4.15}$$

$$> \frac{1}{2}\left(\frac{x}{n+x}\right)^2 \lambda'(n+x) > \left(\frac{x}{n+x}\right)^2 \frac{1}{4\psi(\log(n+x))} > 0\,.$$

Hence, for $n > e^{2K}$

$$u = \frac{1}{2}\left(\frac{x}{n+x}\right)^2 \lambda(n+x) = \frac{x^2}{2(n+x)\psi(\log(n+x))} \tag{4.16}$$

is an increasing function, and we have from (4.14) the following estimation:

$$\sum\nolimits_2 = \sum_{k=m+1}^{\infty} \frac{R_n^k}{R_{n+1} R_{n+2} \ldots R_{n+k}} \tag{4.17}$$

$$< \sum_{k=m+1}^{\infty} \exp\left\{- \frac{k^2}{2(n+k)\psi(\log(n+k))}\right\}$$

$$< \int_m^\infty \exp\left\{-\frac{x^2}{2(n+x)\psi(\log(n+x))}\right\} dx\,.$$

Substituting

$$u = \frac{x^2}{2(n+x)\psi(\log(n+x))}\,; \quad u_m = \frac{m^2}{2(n+m)\psi(\log(n+m))}$$

we obtain that:

$$\sum_2 < \int_{u_m}^{\infty} e^{-u} \frac{dx}{du} \, du .$$

Now, in view of (4.1)

$$\psi(\log(n+x)) \leq \{\log(n+x)\}^{K} \psi(1) < \tfrac{1}{4}(n+x)^{1/2} \tag{4.18}$$

for $n > n_0$ and, hence, by (4.16)

$$u = \left(\frac{x}{n+x}\right)^2 \frac{n+x}{2\psi(\log(n+x))} > 2\left(\frac{x}{n+x}\right)^2 (n+x)^{1/2}, \tag{4.19}$$

$$2(n+x)^{1/2} < \left(\frac{n+x}{x}\right)^2 u \qquad \text{for} \quad n > n_0 .$$

Finally, from (4.15), (4.18) and (4.19)

$$\frac{dx}{du} < \left(\frac{n+x}{x}\right)^2 4\psi(\log n+x)) < \left(\frac{n+x}{x}\right)^2 2(n+x)^{1/2} \tag{4.20}$$

$$< \left(\frac{n+x}{x}\right)^4 u < \left(\frac{n+m}{m}\right)^4 u \qquad \text{for} \quad \begin{matrix} x \geqq m \\ n > n_0 \end{matrix} .$$

Using this estimate, we obtain finally:

$$\sum_2 < \int_{u_m}^{\infty} \left(\frac{n+m}{m}\right)^4 u \cdot e^{-u} \, du = \left(\frac{n+m}{m}\right)^4 (u_m + 1) e^{-u_m}$$

$$= \left(\frac{n+m}{m}\right)^4 \left(1 + \frac{m^2}{2(n+m)\psi(\log(n+m))}\right) \exp\left\{-\frac{m^2}{2(n+m)\psi(\log(n+m))}\right\}$$

wich gives (4.7).

Lemma II.[10] Let $N(r) = N$, $\mu(r)$ and $M(r)$ denote respectively the central index, the maximum-term, and the maximum-modulus of an integral function:

$$f(z) = \sum_{0}^{\infty} c_n z^n$$

(of finite or infinite order).

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10. Somewhat weaker, but more general results of this type were proved by J. Clunie ([3]).

Then, outside a set of finite logarithmic measure, we have the following inequalities:

$$(4.21) \qquad \text{(i)} \quad \frac{N(r)}{\psi(\log N(r))} < 3 \log \mu(r)$$

where $\psi(x)$ is any positive increasing function, defined for $x \geqq 0$, and such that

$$(4.22) \qquad \int_0^\infty \frac{dx}{\psi(x)} < \infty .$$

$$(4.23) \qquad \text{(ii)} \quad \left| \sum_{n \leqq (1-\rho) N} c_n z^n \right| \leqq \mu(r) \exp \left\{ - \frac{\rho^2 N}{3 \psi(\log N)} \right\}$$

for $N > N_0$

$$(4.24) \qquad \left| \sum_{n \geqq (1+\sigma) N} c_n z^n \right| \leqq \mu(r) \exp \left\{ - \frac{\sigma^2}{3(1+\sigma)} \frac{N}{\psi(\log N)} \right\}$$

for $N > N_0(\sigma)$

where $0 < \rho < 1$, $0 < \sigma < \infty$ and $\psi(x)$ is any positive increasing function, defined for $x \geqq 0$, satisfying both (4.22) and the condition:

$$(4.25) \qquad \psi(\lambda x) \leqq \lambda^K \psi(x)$$

for every $x > 0$, $\lambda \geqq 1$, with a suitable K.

$$(4.26) \qquad \text{(iii)} \quad M(r) < N \mu(r)$$

$$(4.27) \qquad \text{(iv)} \quad N(r) > \{1 + o(1)\} \frac{\log M(r)}{\log r}$$

$$(4.28) \qquad \text{(v)} \quad \log N(r) < \{1 + o(1)\} \log \log M(r) .$$

Proof of (i)—(ii).

We define R_n as in Lemma I, and $F(z)$ by:

$$(4.29) \qquad F(z) = \sum_{n=0}^{\infty} \frac{z^n}{R_1 R_2 \dots R_n} .$$

By virtue of (4.22), $\lim R_n = 1$, and hence $F(z)$ is convergent for $|z| < 1$.

If we introduce the functions: $\nu(R, F)$ and $\mu(R, F)$ to denote the central-index and maximum-modulus of $F(z)$, we see immediately, that:

$$\nu(R, F) = N, \quad \text{for } R_N \leq R < R_{n+1} \qquad \text{for } N = 1, 2, 3, \dots .$$

From (4.5) we have that:

$$(4.30) \qquad 3 \log \mu(R_N, F) > \frac{N}{\phi(\log N)}$$

and substituting $n = N$, $m = \rho N$ and $m = \sigma N$ into (4.6) and (4.7) respectively, we obtain further that:

$$(4.31) \qquad \sum_{N-l \leq (1-\rho) N} \frac{R_{N-l+1} \dots R_{N-1} R_N}{R_N^l} < \exp\left\{- \frac{\rho^2 N}{3\phi(\log N)}\right\}$$

$$(4.32) \qquad \sum_{N+k \geq (1+\sigma) N} \frac{R_N^k}{R_{N+1} R_{N+2} \dots R_{N+k}} < \exp\left\{- \frac{\sigma^2}{3(1+\sigma)} \frac{N}{\phi(\log N)}\right\}.$$

To obtain (4.32) we have used (4.25) to prove that:

$$(4.33) \quad \phi(\log(1+\sigma) N) = \phi(\log(N^{1+\frac{\log(1+\sigma)}{\log N}})) = \phi\left\{\left(1 + \frac{\log(1+\sigma)}{\log N}\right) \log N\right\}$$

$$< \exp\left\{\frac{K \log(1+\sigma)}{\log N}\right\} \phi(\log N) = \{1 + o(1)\} \phi(\log N).$$

Applying the Wiman-Valiron Lemma, the inequalities (4.21), (4.23) and (4.24) follow immediately from (4.30), (4.31) and (4.32) respectively.

(iii) Putting $\rho = \sigma = \frac{1}{3}$, (4.26) is an immediate consequence of (4.23) and (4.24).

(iv) Since $c_n \to 0$, we have that:

$$r^N > \mu(r) = c_N r^N, \quad \text{for } N(r) \geqq N_0 .$$

Hence, in view of (4.26)

$$r^N > \mu(r) > \frac{M(r)}{N}$$

$$N \log r > \log M(r) - \log N$$

$$\{1 + o(1)\} N \log r > \log M(r)$$

which proves (4.27).

$$\text{(v)} \qquad \phi(x) = \begin{cases} 1, & \text{if } 0 \leqq x \leqq 1 \\ x^2, & \text{if } 1 \leqq x < \infty \end{cases}$$

satisfies (4.22), hence we obtain from (4.21) as a special case, that:

$$\frac{N}{\log^2 N} < \log \mu(r)$$

$$\log N - 2 \log\log N < \log\log \mu(r)$$

which proves (4.28).

5. Lemma III. Let

$$f(z) = \sum_0^\infty c_n z^n$$

be an integral function, $M(r)$, $\mu(r)$ and $N(r)$ its maximum-modulus, maximal-term and central-index respectively.

We assume that the growth $M(r)$ is subject to the condition:

$$\text{(5.1)} \qquad \overline{\lim_{r\to\infty}} \frac{\log\log\log M(r)}{\log r} = \rho < \infty .$$

We have the following inequalities:

$$\text{(5.2)} \qquad \log|c_n| < -\frac{n}{\lambda} \log\log n \qquad (n = 2, 3, \ldots)$$

$$\text{(5.3)} \qquad \frac{1}{\lambda} \log\log N(r) < \log r$$

for any $\lambda > \rho$, $n > n_0$ and $r > r_0$.

Proof of the Lemma: By Cauchy's inequality:

$$\text{(5.4)} \qquad \log|c_n| \leqq \log M(r) - n \log r < e^{r^{\rho_1}} - n \log r$$

for $\lambda > \rho_1 > \rho$, $r > r_0$.

We write now:

$$\log r_n = \frac{1}{\rho_1} \log\log n ,$$

and substitute $r = r_n$ into (5.4)

$$\log|c_n| \leqq n - \frac{n}{\rho_1} \log\log n < -\frac{n}{\lambda} \log\log n\,, \qquad \text{for } n > n_0\,,$$

which proves (5.2).

Let us write: $\rho < \lambda_1 < \lambda$

$$\log B_n = -\frac{n}{\lambda_1} \log\log n \qquad n = 2, 3, \ldots$$

(5.5) $$\log R_n = \log B_{n-1} - \log B_n\,.$$

Then for: $R_N \leqq r < R_{N+1}$, and every n:

(5.6) $$\log B_n + n \log r \leqq \log B_N + N \log r$$

From (5.5)

$$\log r \geqq \log R_N > \frac{1}{\lambda_1} \log\log N$$

(5.7) $$N < e^{r^{\lambda_1}}$$

(5.8) $$\log r \leqq \log R_{N+1} = \frac{1}{\lambda_1} \log\log N + o(1)\,.$$

From (5.6), (5.7) and (5.8)

$$\log B_n + n \log r \leqq \log B_n + N \log r \leqq -\frac{N}{\lambda_1} \log\log N$$

$$+ \frac{N}{\lambda_1} \log\log N + o(N) = o(N) = o(e^{r^{\lambda_1}})\,.$$

Hence, in view of (5.2) we obtain, for $n > n_0$

$$\log|c_n| + n \log r \leqq \log B_n + n \log r < C\, e^{r^{\lambda_1}}$$

or, writing: $t = \log r$,

(5.9) $$\log|c_N| + Nt < C \cdot e^{e^{\lambda_1 t}}\,.$$

Since $\lim\limits_{r\to\infty} r = \infty$ we find, writing:

$$t_0 = \log r_0\,, \quad N = N(r_0)$$

(5.10) $$\log|c_N| + N t_0 = \log \mu(r_0) > 0$$

for $N > N_0$.

Subtracting (5.10) from (5.9):

$$N(t - t_0) < C\, e^{e^{\lambda_1 t}}.$$

If we write:

$$t_1 = \frac{1}{\lambda_1} \log\log N$$

$$e^{e^{\lambda_1 t_1}} = N$$

we obtain finally, that:

$$t_1 - t_0 < C\,; \quad t_0 = \log r_0 > \frac{1}{\lambda_1} \log\log N - C > \frac{1}{\lambda_1} \log\log N\,,$$

for $N > N_0$, which proves (5.3).

Lemma IV. Let $N = N(r)$, $\mu(r)$ and $M(r)$ denote respectively the central-index, the maximum-term, and the maximum-modulus of an integral function:

$$f(z) = \sum_0^\infty c_n z^n.$$

We assume that $M(r)$ is subject to the growth-restriction:

(5.11) $$\overline{\lim}\ \frac{\log\log\log M(r)}{\log r} = \rho^* < \infty\,.$$

Then for any given: $0 < \varepsilon < 1$ and $\tau > \dfrac{\rho^*}{\varepsilon}$, we have, outside certain exceptional intervals of r, whose upper logarithmic "density" does not exceed ε, the following inequalities:

(5.12) (i) $$\frac{N(r)}{\tau \cdot \log N(r)} < 3 \log \mu(r)$$

(5.13) (ii) $$\left| \sum_{n \leqq (1-\rho)N} c_n z^n \right| \leqq \mu(r) \exp\left\{ - \frac{\rho^2 N}{3\tau \log N} \right\}$$

for $0 < \rho < 1$, and $N > N_0(\rho)$

(5.14) $$\left| \sum_{n \geqq (1+\sigma)N} c_n z^n \right| \leqq \mu(r) \exp\left\{ - \frac{\sigma^2}{3(1+\sigma)} \cdot \frac{N}{\tau \cdot \log N} \right\}$$

for any $0 < \sigma < \infty$, and $N > N_1(\sigma)$.

Proof of the Lemma: We define $\phi(t)$, $\beta(x)$, R_n, R'_n and $F(z)$ by

$$\phi(t) = \begin{cases} \tau, & \text{for} \quad 0 \leqq t \leqq 1 \\ \tau t, & \text{for} \quad 1 \leqq t \end{cases}$$

$$\beta(x) = \int_0^{\log x} \frac{dt}{\phi(t)} = \begin{cases} \dfrac{1}{\tau} \log x, & \text{for} \quad 1 \leqq x \leqq e \\[2ex] \dfrac{1}{\tau}(1 + \log\log x), & \text{for} \quad e \leqq x \end{cases}$$

$$R_n = e^{\beta(n)} = e^{1/\tau} (\log n)^{1/\tau} \qquad \text{for} \quad n = 3, 4, \ldots$$

$$R_1 = e^{\beta(1)} = 1 \ ; \ R_2 = e^{\beta(2)} = 2^{1/\tau}$$

$$R'_n = e^{-1/\tau} R_n$$

$$R'_n = (\log n)^{1/\tau} \qquad \text{for} \quad n = 3, 4, \ldots$$

$$F(z) = \sum_{n=0}^{\infty} \frac{z^n}{R'_1 R'_2 \ldots R'_n} = \sum_0^{\infty} A_n z^n . \tag{5.15}$$

$F(z)$ is an integral function. For the coefficients A_n we find the estimate

$$\log A_n = -\sum_{m=1}^{n} \log R'_m = O(1) - \frac{1}{\tau} \sum_3^n \log\log m \tag{5.16}$$

$$= -\frac{1}{\tau} \int_e^n \log\log x \, dx + O(\log\log n) = -\frac{1}{\tau} n \log\log n$$

$$+ \frac{1}{\tau} \int_e^n \frac{dx}{\log x} + O(\log\log n) = -\frac{n}{\tau} \log\log n + o(n) .$$

We apply now Lemma I. (4.5) gives us the inequality:

$$3 \log \mu(R'_N, F) > \frac{N}{\tau \log N} . \tag{5.17}$$

Further, substituting $n = N$, $m = \rho N$ and $m = \sigma N$ into (4.6) and (4.7) respectively, we obtain the inequalities:

$$\sum_{N-l \leqq (1-\rho) N} \frac{R_{N-l+1} \ldots R_{N-1} R_N}{R_N^l} < \exp\left\{-\frac{\rho^2 N}{3\tau \log N}\right\} \tag{5.18}$$

$$\sum_{N+k \geqq (1+\sigma) N} \frac{R_N^k}{R_{N+1} R_{N+2} \ldots R_{N+k}} < \exp\left\{-\frac{\sigma^2}{3(1+\sigma)} \frac{N}{\tau \log N}\right\} . \tag{5.19}$$

Since (5.11) is satisfied, the inequality (5.2) is valid for $\lambda = \varepsilon\tau > \rho^*$:

(5.20) $$\log |c_n| < -\frac{n}{\varepsilon\tau} \log\log n \qquad \text{for} \quad n > n_0 .$$

Comparing (5.20) and (5.16) we see that:

$$\varphi(\zeta) = \sum_{0}^{\infty} \frac{c_n}{A_n} \zeta^n$$

is an integral function. Hence we can legitimately apply the Wiman-Valiron lemma to the function $f(z)$, with the comparison-function $F(z)$. The inequalities (5.12), (5.13) and (5.14) can now be derived immediately from the inequalities (5.17), (5.18) and (5.19).

It only remains to discuss the exceptional intervals. The logarithmic measure of the exceptional interval $(r'_{n_k}, r_{n_{k+1}})$ is:

$$l_k = \int_{r'_{n_k}}^{r_{n_{k+1}}} \frac{dr}{r} = \log \frac{r_{n_{k+1}}}{r'_{n_k}}$$

and by virtue of (3.7)

(5.21) $$\sum_{k=1}^{m-1} l_k = \sum_{k=1}^{m-1} \log \frac{r_{n_{k+1}}}{r'_{n_k}} = \log R'_{n_m} - \log R'_{n_1}$$

$$= \frac{1}{\tau} \log\log n_m + O(1) .$$

On the other hand, applying the inequality (5.3) with

$$\lambda = \varepsilon\tau > \rho^* \quad \text{and} \quad r = r_{n_m}$$

$$\log r_{n_m} > \frac{1}{\varepsilon\tau} \log\log n_m$$

which gives for the logarithmic measure of the whole segment (r_{n_1}, r_{n_m}), the estimate

(5.22) $$L_{m-1} = \int_{r_{n_1}}^{r_{n_m}} \frac{dr}{r} = \log r_{n_m} - \log r_{n_1} < \frac{1}{\varepsilon\tau} \log\log n_m + O(1) .$$

Combining (5.21) with (5.22) we deduce that

$$\limsup_{m\to\infty} \frac{l_1 + l_2 + \dots + l_{m-1}}{L_{m-1}} \leq \varepsilon \tag{5.23}$$

which establishes our assertion about the upper logarithmic density of the exceptional intervals.

Lemma V. Let $f(z)$ be an integral function of finite order (at most) ρ. Then (using the same notation as in Lemma IV) for any given $0 < \varepsilon < 1$, and $\tau > \frac{\rho}{\varepsilon}$, we have, outside certain exceptional intervals of r, whose upper logarithmic density does not exceed ε, the following inequality

$$N(r) < \tau \log \mu(r). \tag{5.24}$$

The proof[11] of this lemma is completely analogous to that of the first part of Lemma IV.

Lemma VI.[12] For any integral function $f(z)$ we have, outside an exceptional set of finite logarithmic measure, the following asymptotic formula

$$f(z_0 e^{it}) = e^{iNt} f(z_0) \{1 + o(1)\} \tag{5.25}$$

if

$$|z_0| = r, \quad |f(z_0)| = M(r) \quad \text{and} \quad |t| < N^{-15/16}.$$

Here $N = N(r)$ and $M(r)$ are defined as in Lemma II.

6. Lemma VII. If

$$f(z) = \sum_{n=1}^{\infty} a_n z^{\lambda_n}$$

is an integral function, satisfying the gap-condition

$$\lambda_{n+1} - \lambda_n \geqq 2\Lambda(\lambda_n) \qquad \text{for} \quad \lambda_n > N_0 \tag{6.1}$$

where $\Lambda(x)$ is a positive increasing function, and such that

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11. Kövari [5], p. 324.

12. Valiron [12] p. 101—102. Stronger results of this type were proved by J. Clunie [3].

(6.2) $$\lim_{x \to \infty} \Lambda(x) = \infty, \quad \Lambda(kx) \leq k\Lambda(x)$$

for every $x \geqq x_0, \quad k \geqq 1,$

then the inequality

(6.3) $$\int_{\Theta_r - \frac{\delta_r}{2}}^{\Theta_r + \frac{\delta_r}{2}} |f(re^{i\vartheta})|^2 \, d\vartheta \geqq \{1 - o(1)\} \frac{\delta_r}{32\pi} \int_0^{2\pi} |f(re^{i\vartheta})|^2 \, d\vartheta$$

holds outside an exceptional set of finite logarithmic measure, provided that

(6.4) $$\delta_r \geqq \frac{16\pi}{\Lambda(N(r))}$$

where $N(r)$ is the central-index of $f(z)$. (Otherwise Θ_r and δ_r are arbitrary functions of r.)

Proof of the Lemma:

We assume that $N = N(r) > 2N_0$, and we write

$$f(z) = G(z) + H(z)$$

(6.5) $$G(z) = \sum_{\lambda_n \leq N/2} a_n z^{\lambda_n} \quad ; \quad H(z) = \sum_{\lambda_n > N/2} a_n z^{\lambda_n} .$$

Denoting the maximal term of $f(z)$ by $\mu(r)$, we have by virtue of (4.23), that

$$|G(re^{i\vartheta})| = o\left(\frac{\mu(r)}{N(r)}\right)$$

$$\int_{\Theta - \delta/2}^{\Theta + \delta/2} |G(re^{i\vartheta})|^2 \, d\vartheta \leqq \int_0^{2\pi} |G(re^{i\vartheta})|^2 \, d\vartheta = o\left(\frac{\mu(r)^2}{N(r)^2}\right)$$

outside an exceptional set.

For $\lambda_n > \dfrac{N}{2}$ we find that

$$\lambda_{n+1} - \lambda_n > 2\Lambda(\lambda_n) \geqq 2\Lambda\left(\frac{N}{2}\right) \geqq \Lambda(N) \geqq \frac{16\pi}{\delta} .$$

Hence we can apply "Wiener's lemma" to $\varphi(t) = H(re^{it})$ with $L = 16\pi/\delta$,

and obtain that

$$(6.7)\qquad \int_{\Theta-\delta/2}^{\Theta+\delta/2} |H(re^{i\vartheta})|^2\, d\vartheta \geq \frac{\delta}{32\pi}\int_0^{2\pi} |H(re^{i\vartheta})|^2\, d\vartheta\,.$$

Finally we have the inequality

$$(6.8)\qquad \int_0^{2\pi} |f(re^{i\vartheta})|^2\, d\vartheta = 2\pi\sum_1^{\infty}|a_n|^2 r^{2\lambda_n} \geqq 2\pi\mu(r)^2\,.$$

In view of the inequalities (6.6—8) we obtain now, by means of Minkowsky's inequality, that

$$\sqrt{\int_{\Theta-\delta/2}^{\Theta+\delta/2} |f|^2} + \sqrt{\int_{\Theta-\delta/2}^{\Theta+\delta/2} |G|^2} \geq \sqrt{\int_{\Theta-\delta/2}^{\Theta+\delta/2} |H|^2}$$

$$\geqq \sqrt{\frac{\delta}{32\pi}}\cdot\sqrt{\int_0^{2\pi}|H|^2} \geqq \sqrt{\frac{\delta}{32\pi}}\left\{\sqrt{\int_0^{2\pi}|f|^2}-\sqrt{\int_0^{2\pi}|G|^2}\right\}$$

$$\sqrt{\int_{\Theta-\delta/2}^{\Theta+\delta/2}|f|^2} \geqq \sqrt{\frac{\delta}{32\pi}}\cdot\sqrt{\int_0^{2\pi}|f(re^{i\vartheta})|^2} + o(\mu(r)\,\delta_r)$$

$$= \{1+o(1)\}\sqrt{\frac{\delta}{32\pi}\int_0^{2\pi}|f|^2}$$

which proves the lemma.

Lemma VIII.[13] Let

$$f(z) = \sum_{n=1}^{\infty} a_n z^{\lambda_n}$$

be an integral function, and

$$(6.9)\qquad M(r) = \max_{|z|=r} |f(re^{i\vartheta})|$$

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13. This result was first published—in a slightly weaker form—by the author in [5]. It generalises a result of Turán [9], who proved

$$\overline{\lim_{r\to\infty}}\,\frac{\log M(r,\Theta,\delta)}{\log M(r)} = 1$$

under the gap-condition:

$$\frac{\log\lambda_n}{\log n} \geqq 1+h>1\,.$$

$$M(r, \Theta, \delta) = \max_{|\vartheta-\Theta| \leqq \delta/2} |f(re^{i\vartheta})| . \tag{6.10}$$

Suppose that $f(z)$ satisfies the gap-condition:

$$\frac{\lambda_n}{n\psi(\log n)\Delta(n)} \tag{6.11}$$

where $\psi(x)$ is a positive increasing function, satisfying the conditions:

$$\int_0^{\infty} \frac{dx}{\psi(x)} < \infty \tag{6.12}$$

$$\psi(\lambda x) \leqq \lambda^K \psi(x) \tag{6.13}$$

with a suitable constant K, for any $x>0$, $\lambda \geqq 1$ and $\Delta(x)$ is a positive increasing function, such that:

$$\Delta(x^\lambda) \leqq \lambda\Delta(x) \tag{6.14}$$

for any $x>x_0$, $\lambda \geqq 1$.

Then

$$\frac{\log M(r, \Theta_r, \delta_r)}{\log M(r)} \to 1 \tag{6.15}$$

outside an exceptional set of finite logarithmic measure, provided that:

$$\delta_r \geqq e^{-\Delta(N(r))} \tag{6.16}$$

where $N(r)$ is the central-index of $f(z)$. (Otherwise Θ_r and δ_r are arbitrary functions of r.)

Proof of the Lemma: Since $\psi(x)$ satisfies the conditions of Lemma II, the inequalities:

$$\frac{N(r)}{\psi(\log N(r))} < 3 \log M(r) \tag{6.17}$$

$$\left| \sum_{\lambda_n > 2N} a_n z^{\lambda_n} \right| < M(r) \exp\left\{ -\frac{N}{6\psi(\log N)} \right\} \tag{6.18}$$

are satisfied outside an exceptional set of finite logarithmic measure.

We first show that (6.11) implies:

$$\frac{\lambda_n}{n\psi(\log\lambda_n)\,\Delta(\lambda_n)} \to \infty\,. \tag{6.19}$$

Indeed, suppose that:

$$\frac{\lambda_{n_k}}{n_k\,\psi(\log\lambda_{n_k})\,\Delta(\lambda_{n_k})} < A \qquad \text{for} \quad k = 1, 2, 3, \ldots \tag{6.20}$$

$$\log\lambda_{n_k} < \log A + \log n_k + \log\psi(\log\lambda_{n_k}) + \log\Delta(\lambda_{n_k})\,.$$

In view of (6.13–14)

$$\psi(\log\lambda_{n_k}) \leqq (\log\lambda_{n_k})^K\,\psi(1)$$

$$\log\psi(\log\lambda_{n_k}) \leqq K\log\log\lambda_{n_k} + \log\psi(1) = o(\log\lambda_{n_k})$$

$$\Delta(\lambda_{n_k}) \leqq \frac{\Delta(x_0)\log\lambda_{n_k}}{\log x_0}$$

$$\log\Delta(\lambda_{n_k}) = o(\log\lambda_{n_k})\,.$$

Hence

$$\log\lambda_{n_k} < \{1 + o(1)\}\log n_k$$

$$\psi(\log\lambda_{n_k}) < \psi\{(1 + o(1))\log n_k\} \leqq \{1 + o(1)\}\,\psi(\log n_k) \tag{6.21}$$

$$\Delta(\lambda_{n_k}) = \Delta(e^{\log\lambda_{n_k}}) \tag{6.22}$$

$$\leqq \Delta(e^{\{1+o(1)\}\log n_k}) = \Delta(n_k^{1+o(1)}) = \{1 + o(1)\}\,\Delta(n_k)\,.$$

Substituting (6.21) and (6.22) into (6.20)

$$\frac{\lambda_{n_k}}{n_k\,\psi(\log n_k)\,\Delta(n_k)} < \{1 + o(1)\}\,A$$

which contradicts (6.11).

If $$\lambda_l \leqq 2N < \lambda_{l+1}$$

then by virtue of (6.19)

$$l = \frac{o(\lambda_l)}{\psi(\log\lambda_l)\,\Delta(\lambda_l)} = \frac{o(2N)}{\psi(\log 2N)\cdot\Delta(2N)} \tag{6.23}$$

$$= \frac{o(N)}{\psi(\log N)\,\Delta(N)}$$

since, in view of (6.13—14)

$$\frac{x}{\psi(\log x)\,\Delta(x)}$$

is an increasing function for $x \geqq x_0$.

We apply now "Turán's lemma", and obtain

$$\max_{|z|=r}\left|\sum_{\lambda_n \leqq 2N} a_n z^{\lambda_n}\right| = \max_{|z|=r}\left|\sum_{n=1}^{l} a_n z^{\lambda_n}\right| \tag{6.24}$$

$$\leqq \exp\left\{\left(\log\frac{48\pi}{\delta}\right) l\right\} \cdot \max_{\substack{|z|=r \\ \Theta-\delta/2 \leqq \arc z \leqq \Theta+\delta/2}}\left|\sum_{n=1}^{l} a_n z^{\lambda_n}\right|$$

$$= \exp\left\{\frac{o(N)}{\psi(\log N)}\right\} \cdot \max_{\substack{|z|=r \\ \Theta-\delta/2 \leqq \arc z \leqq \Theta+\delta/2}}\left|\sum_{\lambda_n \leqq 2N} a_n z^{\lambda_n}\right|$$

by virtue of (6.16) and (6.23). Combining (6.24) with (6.18) we obtain

$$M(r) \leqq \max_{|z|=r}\left|\sum_{\lambda_n \leqq 2N} a_n z^{\lambda_n}\right| + \max_{|z|=r}\left|\sum_{\lambda_n > 2N} a_n z^{\lambda_n}\right|$$

$$\leqq \exp\left\{\frac{o(N)}{\psi(\log N)}\right\} \max_{\substack{|z|=r \\ \Theta-\delta/2 \leqq \arc z \leqq \Theta+\delta/2}}\left|\sum_{\lambda_n \leqq 2N} a_n z^{\lambda_n}\right| + o(M(r))$$

$$\leqq \exp\left\{\frac{o(N)}{\psi(\log N)}\right\}\left[M(r,\Theta,\delta) + \max_{|z|=r}\left|\sum_{\lambda_n > 2N} a_n z^{\lambda_n}\right|\right] + o(M(r))$$

$$= \exp\left\{\frac{o(N)}{\psi(\log N)}\right\}\left[M(r,\Theta,\delta) + M(r)\exp\left\{-\frac{N}{6\psi(\log N)}\right\}\right] + o(M(r))$$

$$= \exp\left\{\frac{o(N)}{\psi(\log N)}\right\} M(r,\Theta,\delta) + o(M(r))$$

$$\{1 + o(1)\}\, M(r) \leqq \exp\left\{\frac{o(N)}{\psi(\log N)}\right\} M(r,\Theta,\delta)$$

$$o(1) + \log M(r) \leqq \frac{o(N)}{\psi(\log N)} + \log M(r,\Theta,\delta)$$

$$= o(\log M(r)) + \log M(r,\Theta,\delta)$$

by virtue of (6.17). Thus we obtained, that

$$\log M(r,\Theta,\delta) = \{1 + o(1)\}\log M(r)$$

which is the desired result.

Lemma IX. Suppose that

$$f(z) = \sum_{n=1}^{\infty} a_n z^{\lambda_n}$$

is an integral function, which satisfies the gap-condition

$$\frac{\lambda_n}{n \cdot \log n \cdot \Delta(n)} \to \infty \tag{6.25}$$

and is subject to the growth-restriction

$$\overline{\lim_{r\to\infty}} \frac{\log\log M(r)}{\log r} = \rho^* < \infty\,. \tag{6.26}$$

Then

$$\frac{\log M(r\,,\,\Theta_r\,,\,\delta_r)}{\log Mr} \to 1 \tag{6.27}$$

outside an exceptional set of zero logarithmic "density", provided that

$$\delta_r \geqq e^{-\Delta(N(r))}\,. \tag{6.28}$$

We assume here that $M(r)$ and $M(r\,,\Theta\,,\delta)$ are defined by (6.9) and (6.10) respectively, and that $\Delta(x)$ is subject to the same conditions as in the previous lemma.

Proof of the Lemma: We first note that the only use we have made of condition (6.12) in the proof of Lemma VIII was to establish the inequalities (6.17—18). Consequently the conclusions of the lemma remain true, if we omit (6.12), but assume that the inequalities (6.17—18) are nevertheless valid outside a certain exceptional set.

Now, by virtue of the condition (6.26) and Lemma IV, if $\tau > \frac{\rho^*}{\varepsilon}$ the above inequalities are in fact true with

$$\psi(x) = \tau x$$

outside certain exceptional intervals of r, whose upper logarithmic density does not exceed ε. Since ε can be arbitrarily small, this proves our lemma.

Lemma X. Given the positive increasing functions: $v(x)$, $l(x)$ and $\psi(y)$ satisfying

$$\lim_{x\to\infty} \nu(x) = \lim_{x\to\infty} l(x) = \infty \tag{6.29}$$
$$\phi(\lambda y) \leqq \lambda^K \phi(y)$$

for every $y>0$ and $\lambda>1$, the inequality

$$\frac{\nu(x)}{\phi(\log \nu(x))} < A \cdot l(x) \tag{6.30}$$

will always imply the inequality

$$\nu(x) < 2A \cdot l(x)\, \phi(\log l(x)) \tag{6.31}$$

for all sufficiently large x.

Proof of the Lemma: We substitute $\lambda = t$, $y = 1$ into (6.29)

$$\phi(t) \leqq t^K \phi(1) \;;\; \log \phi(t) \leqq K \log t + \log \phi(1). \tag{6.32}$$

From (6.30) and (6.32) we deduce that

$$\log A + \log l(x) > \log \nu(x) - \log \phi(\log \nu(x))$$
$$> \log \nu(x) - K \log\log \nu(x) - \log \phi(1)$$

$$\log l(x) > \log \nu(x) \{1 + o(1)\}$$
$$\log \nu(x) < \{1 + o(1)\} \log l(x).$$

Using (6.29) again, we obtain

$$\phi(\log \nu(x)) \leqq \phi\{[1 + o(1)] \log l(x)\} \tag{6.33}$$
$$\leqq \{1 + o(1)\}\, \phi(\log l(x)).$$

From (6.30) and (6.33) we obtain finally the desired result

$$\nu(x) < A\{1 + o(1)\}\, l(x)\, \phi(\log l(x)).$$

7. We proceed now to discuss the situation referred to in Theorems 1—4. That is we consider an integral function $f(z)$ of infinite order which has $z = 0$ as a Borel-exceptional value (in the sense of Theorem 1). If ρ is the exponent of convergence of the zeros of $f(z)$, then by a classical result of Hadamard, $f(z)$ can be written in the form

$$f(z) = P(z)\, e^{g(z)} \tag{7.1}$$

where $P(z)$ is a canonical product of order ρ, and $g(z)$ an integral function. Let $M(r)$, $M_1(r)$, $M_2(r)$ and $m(r)$ denote the maximum-modulus

of $f(z)$, $P(z)$, $\frac{1}{P(z)}$ and $g(z)$ respectively; $N(r)$ and $\nu(r)$ the central index of $f(z)$ and $g(z)$ respectively.

Finally we write

$$\max_{|t-\Theta_1|\leqq\delta/2} |f(re^{it})| = M(r, \Theta, \delta).$$

The formulas of this paragraph are understood to be valid only outside an exceptional set. These exceptional sets are either of finite logarithmic measure, or of zero logarithmic density, or of 'small' logarithmic density.

Under these conditions, and with the above notations, the following lemmas hold:

Lemma XI.

(7.2) (i) $\lim\limits_{r\to\infty} \frac{m(r)}{r^n} = \infty$, for every n

(7.3) (ii) $m(r) = \log M(r)\{1 + o(1)\}$

(iii) For each r, there exists a Θ_2 such that

(7.4) $$\log M(r, \Theta_2, \delta) \leqq -\{\tfrac{1}{2} + o(1)\}\, m(r)$$

for

(7.5) $$\delta = \frac{2\pi}{3}\,\frac{1}{\nu(r)}.$$

Lemma XII. For any $\psi(x)$, satisfying (4.22)

(7.6) $$\nu(r) < 7 \log N(r)\, \psi(\log\log N(r)).$$

If $g(z)$ is of finite order at most ρ, (7.6) can be replaced by

(7.7) $$\nu(r) < 12\rho \log N(r).$$

Lemma XIII. Suppose that the inequality

(7.8) $$\nu(r) \leqq \frac{1}{24}\Lambda(N(r))$$

is true outside an exceptional set, where $\Lambda(x)$ is a positive increasing function, and such that

(7.9) $$\Lambda(kx) \leqq k\Lambda(x)$$

for every $x \geqq x_0$, $k \geqq 1$.

Then, for infinitely many n, we must have

$$\lambda_{n+1} - \lambda_n < 2\Lambda(\lambda_n). \tag{7.10}$$

Proof of Lemma XI.

(i) Since $f(z)$ is of infinite, and $P(z)$ of finite order, $g(z)$ must be a transcendental integral function. Hence (7.2).

(ii)—(iii) If $|g(re^{i\vartheta})|$ attains its maximum for $\vartheta = \varphi$, we have that

$$g(re^{i\varphi}) = m(r)\, e^{i\psi}$$

and by Lemma VI

$$g(re^{i(\varphi+t)}) = m(r)\, e^{i(\psi+\nu t)}\{1 + o(1)\} \qquad \text{for } t = O(1/\nu).$$

We write now

$$\Theta_1 = \varphi - \frac{\psi}{\nu} \quad \text{and} \quad \Theta_2 = \Theta_1 + \frac{\pi}{\nu}.$$

Then we have for $\tau = O\left(\frac{1}{\nu}\right)$

$$\begin{aligned} g(re^{i(\Theta_1+\tau)}) &= m(r)\, e^{i\nu\tau}\{1 + o(1)\} \\ g(re^{i(\Theta_2+\tau)}) &= -\, m(r)\, e^{i\nu\tau}\{1 + o(1)\} \end{aligned} \tag{7.11}$$

$$\begin{aligned} \operatorname{Re}\{g(re^{i(\Theta_1+\tau)})\} &= m(r)\{\cos\nu\tau + o(1)\} \\ \operatorname{Re}\{g(re^{i(\Theta_2+\tau)})\} &= -m(r)\{\cos\nu\tau + o(1)\}. \end{aligned} \tag{7.12}$$

Since $P(z)$ is of order ρ, we have not only

$$\log M_1(r) = O(r^{\rho+1}) \tag{7.13}$$

but, by a classical result of Borel, also

$$\log M_2(r) = O(r^{\rho+1}); \tag{7.14}$$

((7.14) of course holds only outside an exceptional set). Now

$$\log|f(re^{i\vartheta})| = \log|P(re^{i\vartheta})| + \operatorname{Re}\{g(re^{i\vartheta})\},$$

hence, by virtue of (7.12—14) and (7.2) we have

$$\log M(r) < \log M_1(r) + m(r) = O(r^{\rho+1}) + m(r) = \{1 + o(1)\}\, m(r) \tag{7.15}$$

$$\begin{aligned} \log M(r) &\geq \log|f(re^{i\Theta_1})| = \log|P(re^{i\Theta_1})| + \operatorname{Re}\{g(rc^{i\Theta_1})\} \\ &\geq O(r^{\rho+1}) + m(r)\{1 + o(1)\} = \{1 + o(1)\}\, m(r) \end{aligned} \tag{7.16}$$

$$\begin{aligned}(7.17)\qquad \log|f(re^{i(\theta_2+\tau)})| &= O(r^{\rho+1}) - m(r)\{\cos\nu\tau + o(1)\}\\ &= -m(r)\{\cos\nu\tau + o(1)\} \leqq -\{\tfrac{1}{2} + o(1)\}\, m(r) \qquad \text{for } |\tau| \leqq \pi/3\nu.\end{aligned}$$

The combination of (7.15) and (7.16) gives (7.3) while (7.4) is an immediate consequence of (7.17).

Proof of Lemma XII.

Using (4.27) and (7.3) we have the inequality

$$N(r) > \{1 = o(1)\}\frac{\log M(r)}{\log r} = \{1 + o(1)\}\frac{m(r)}{\log r}.$$

In view of (7.2) this gives

$$(7.18)\qquad \log N(r) > o(1) + \log m(r) - \log\log r = \{1 + o(1)\}\log m(r).$$

On the other hand, from (4.21)

$$(7.19)\qquad \frac{\nu(r)}{\psi(\log\nu(r))} < 3\log m(r).$$

Combining (7.18) and (7.19) we obtain

$$(7.20)\qquad \frac{\nu(r)}{\psi(\log\nu(r))} < \{3 + o(1)\}\log N(r)$$

and by virtue of Lemma X.

$$\nu(r) < \{6 + o(1)\}\log N(r)\,\psi(\log\log N(r)).$$

This proves (7.6). If $g(z)$ is of finite order (at most) ρ, then we can apply Lemma V, and (writing $\tau = 11\rho$) obtain (outside a set whose upper logarithmic density does not exceed $\frac{1}{10}$)

$$(7.21)\qquad \nu(r) < 11\rho\log m(r).$$

Combining (7.18) and (7.21) we obtain (7.7).

Proof of Lemma XIII.

If we write

$$\delta^r = \frac{2\pi}{3}\,\frac{1}{\nu(r)} \geqq \frac{16\,\pi}{\Lambda(N(r))}$$

then, from (7.4) and (7.3), using the same notations, we obtain

$$\int_{\theta_2-\delta/2}^{\theta_2+\delta/2} |f(re^{i\vartheta})|^2 \, d\vartheta \leqq \delta \cdot e^{-m(r)\{1+o(1)\}}$$

$$\int_0^{2\pi} |f(re^{i\vartheta})|^2 \, d\vartheta > \int_{\theta_1-\delta/2}^{\theta_1+\delta/2} |f(re^{i\vartheta})|^2 \, d\vartheta$$

$$\geqq \delta \cdot e^{m(r)\{1+o(1)\}}$$

$$\log \int_0^{2\pi} |f(re^{i\vartheta})|^2 \, d\vartheta - \log \int_{\theta_2-\delta/2}^{\theta_2+\delta/2} |f(re^{i\vartheta})|^2 \, d\vartheta$$

$$\geqq \{2+o(1)\}\, m(r) = \{2+o(1)\} \log M(r) \,. \tag{7.22}$$

Suppose now that, contrary to our assertion

$$\lambda_{n+1} - \lambda_n \geqq 2\Lambda(\lambda_n)$$

for $\lambda_n > N_0$. Then, by virtue of Lemma VII, and the assumptions of our lemma we have that

$$\log \int_0^{2\pi} |f(re^{i\vartheta})|^2 \, d\vartheta - \log \int_{\theta_2-\delta/2}^{\theta_2+\delta/2} |f(re^{i\vartheta})|^2 \, d\vartheta$$

$$\leqq o(1) + \log \frac{32\pi}{\delta} = o(1) + \log 48\nu(r) \leqq O(1) + \log \Lambda(N(r))$$

$$\leqq O(1) + \log \left\{ \frac{N(r)}{x_0} \Lambda(x_0) \right\} = O(1) + \log N(r) \,. \tag{7.23}$$

Comparing (7.22) and (7.23) we obtain

$$\log N(r) \geqq \log M(r)$$

which, in view of (4.28) is obviously impossible.

Having obtained these preliminary results, we can prove now our theorems easily. In view of Lemma XII, (7.8) is satisfied for

$$\Lambda(x) = 168 \log x \, \psi(\log\log x) \tag{7.24}$$

assuming that $\psi(x)$ satisfies (4.22). If $\psi(x)$ also satisfies (4.25), then $\Lambda(x)$ satisfies (7.3). Hence the conditions of Lemma XIII are satisfied for the function defined by (7.24). We can conclude now, that for infinitely many n

$$\frac{\lambda_{n+1} - \lambda_n}{\log \lambda_n \, \psi(\log \log \lambda_n)} < 336$$

which, writing $\psi(\log x) = \Omega(x)$ proves Theorem I.

If the growth of $f(z)$ is restricted by (2.5) then — in view of (7.3) — $g(z)$ is of finite order (at most) ρ. Then, again by Lemma XII, (7.8) and (7.9) are satisfied for

$$\Lambda(x) = 288 \, \rho \log x .$$

Hence we can apply Lemma XIII, and conclude that for infinitely many n

$$\frac{\lambda_{n+1} - \lambda_n}{\log \lambda_n} < 576 \, \rho$$

which proves Theorem III.

Let now $\Theta_2(r)$ and $\delta(r)$ have the same meaning as in Lemma XI. Then from (7.3) and (7.4) we have that (outside an exceptional set)

$$\frac{\log M(r, \Theta_2, \delta)}{\log M(r)} \leqq -\frac{1}{2} + o(1). \tag{7.25}$$

On the other hand, in view of (7.6)

$$\delta_r = \frac{2\pi}{3} \, \frac{1}{\nu(r)} > \frac{1}{\log^2 N}$$

and hence (6.16) is satisfied for

$$\Delta(x) = 2 \log \log x .$$

This obviously satisfies (6.14).

Choosing any $\psi(x)$ satisfying (6.12) and (6.13) we see that all the assumptions of Lemma VIII will be satisfied if it is assumed that

$$\frac{\lambda_n}{n \psi(\log n) \log \log n} \to \infty . \tag{7.26}$$

We can then conclude that:

$$\frac{\log M(r, \Theta_2, \delta)}{\log M(r)} \to 1 \tag{7.27}$$

(again outside an exceptional set). Since (7.27) obviously contradicts (7.25), Theorem II is proved if we write: $\psi(\log x) = \Omega(x)$.

If $f(z)$ is restricted by the growth-condition (2.5), then we can use Lemma IX instead of Lemma VIII. In fact all the conditions of this lemma will be fulfilled if it is assumed that

$$\frac{\lambda_n}{n \log n \log \log n} \to \infty .$$

We can then again derive (7.27), which contradicts (7.25). This proves Theorem 4.

8. It is clear, that the definition given in paragraph 2 for the Borel exceptional value of an integral function of infinite order is unnecessarily restrictive, and it is easy to see how one can get a more general definition. According to a classical result of Weierstrass, any integral function $f(z)$ has the following (non unique) representation

$$f(z) = P(z)\, e^{g(z)} \tag{8.1}$$

where $P(z)$ is an infinite product. If we write

$$M(r) = \max_{|z|=r} |f(z)|$$

$$M_1(r) = \max_{|z|=r} |P(z)|$$

$$m_1(r) = \min_{|z|=r} |P(z)|$$

and analyse the proof of Lemma XI, we arrive at the following natural definition. [14]

If the integral function $f(z)$ has a representation of the form (8.1) which has the property that

$$\log M_1(r) = o(\log M(r)) \tag{8.2}$$

$$-\log m_1(r) = o(\log M(r)) \tag{8.3}$$

hold simultaneously on a set of positive logarithmic density, then $z = 0$ is said to be a (generalised) Borel exceptional value of $f(z)$. It is clear from the analysis of Lemma XI, that Theorems 1—4 remain true if Borel exceptional values are understood in this, wider sense.

14. Further generalisation — e.g. one involving the Nevanlinna characteristic of $P(z)$ — would be possible, but would not suit our purpose.

Hayman [15] has proved, that for any integral function $P(z)$

$$-\log m_1(r) < 3 \log M_1(r) \log\log\log M_1(r)$$

holds on a set of positive logarithmic density. In view of this result conditions (8.2) and (8.3) can be replaced by the single condition

$$\log M_1(r) \log\log\log M_1(r) = o(\log M(r)) \,. \tag{8.4}$$

The following theorem — the proof of which is omitted — gives (in terms of the zeros of $f(z)$) a sufficient (but far from necessary) condition for $z = 0$ to be a (generalised) Borel exceptional value of $f(z)$,

Theorem 5. *Let $n(r)$ denote the number of zeros of the integral function $f(z)$ in the disc: $|z| \leqq r$, and write*

$$\log n^*(r) = \log r \max_{2 \leqq \rho \leqq r} \frac{\log n(\rho)}{\log \rho} \tag{8.5}$$

$$N(r) = n^*(r^3) + n^*(2r)^3 \log\log n^*(2r) \tag{8.6}$$

$z = 0$ is a (generalised) Borel exceptional value of $f(z)$ if

$$N(r) \log\log N(r) = o(\log M(r)) \,. \tag{8.7}$$

We observe, that if $f(z)$ is of infinite order, and $z = 0$ is a Borel exceptional value (in the original, narrower sense) of $f(z)$ (i.e. if the sequence of zeros has a finite exponent of convergence: ρ) then the condition of Theorem 5 is always satisfied. In this case

$$n(r) = O(r^{\rho+1})$$

$$n^*(r) = O(r^{\rho+1})$$

$$N(r) = O({}^{3\rho+4})$$

$$N(r) \log\log N(r) = O(r^{3\rho+5}) \,. \tag{8.8}$$

We can use the inequalities (7.13) and (7.14) (replacing in (7.14) $\log M_2(r)$ by $-\log m_1(r)$ owing to the change in notation) and write

$$\log M_1(r) = O(r^{\rho+1})$$

$$-\log m_1(r) = O(r^{\rho+1})$$

$$\operatorname{Re}\{g(z)\} > \log|f(z)| - \log M_1(r) = \log|f(z)| - O(r^{\rho+1}) \,.$$

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15. Hayman [11].

Hence $g(z)$ is transcendental, and for $n > \rho + 1$

$$\frac{\log M(r)}{r^n} \geq r^n \max_{|z|=r} \operatorname{Re}\{g(z)\} + o(1) \to \infty \quad \text{as } r \to \infty . \tag{8.9}$$

(8.7) follows now from (8.8) and (8.9).

9. The method developed in paragraphs 4—7 can be easily adapted to establish the following (admittedly rather special) result, which we state here without proof.

Theorem 6. *If the function:* $f(z) = \Sigma a_n z^{\lambda_n}$ *is of the form*

$$f(z) = e^{e^{g(z)}} \tag{9.1}$$

(here $g(z)$ is a transcendental [16] integral function) *we must have*

$$\underline{\lim} \frac{\lambda_{n+1} - \lambda_n}{(\log \log \lambda_n)^{1+\varepsilon}} < \infty \tag{9.2}$$

for any $\varepsilon > 0$.

10. We think it is not impossible that the Fejer-Biernacky result is essentially best possible in the sense that the gap-condition (1.1) can not be replaced by any weaker one, unless we restrict the growth of the function. However it is not even known whether the gap-condition

$$\lambda_{n+1} - \lambda_n \to \infty \tag{10.1}$$

or the slightly weaker condition

$$\frac{\lambda_n}{n} \to \infty \tag{10.2}$$

(both are referred to indiscriminately as the Fabry gap-condition) is compatible or not with the existence of an exceptional value. The only result in this direction is due to W. K. Hayman, who proved that integral functions satisfying

$$\overline{\lim} (\lambda_{n+1} - \lambda_n) = \infty \tag{10.3}$$

16. In the case when $g(z)$ is a polynomial one could easily prove that $\underline{\lim} (\lambda_{n+1} - \lambda) < \infty$.

or even the stronger

$$\overline{\lim}\,\frac{\lambda_n}{n} = \infty \tag{10.4}$$

can have exceptional values. Since this result has not yet been published we reproduce it here.

Given any strictly increasing sequence: $\{N_m\}$ of positive integers, we shall construct an integral function

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \tag{10.5}$$

having no zeros, and with the property, that there exists a subsequence $\{m_k\}$ of natural integers, such that

$$c_n = 0\,, \quad \text{whenever} \quad N_{m_{2k-1}} < n \leqq N_{m_{2k}} \quad \text{for any } k > 0\,. \tag{10.6}$$

Let $\{R_k\}$ be an increasing sequence of positive numbers such that

$$R_0 = 1\,, \qquad \lim_{k\to\infty} R_k = \infty \tag{10.7}$$

and $\{M_k\}$ be a decreasing sequence of positive numbers such that

$$M_0 = 1\,, \qquad \sum_{k=0}^{\infty} M_k < \infty\,. \tag{10.8}$$

We shall construct a sequence of polynomials: $\{P_i(z)\}_0^\infty$ and a sequence: $\{m_i\}_1^\infty$ of positive integers with the following properties:

$$|P_i(z)| \leqq M_i \quad \text{for} \quad |z| \leqq R_i \tag{10.9}$$

$$e^{\sum_{i=1}^{s} P_i(z)} = \sum_{n=0}^{\infty} c_n^{(s)} z^n \tag{10.10}$$

$$c_n^{(s)} = 0 \qquad \text{for} \qquad N_{m_{2j-1}} < n \leqq N_{m_{2j}} \qquad j = 1, 2, \dots, s$$

$$c_n^{(s)} = c_n^{(s+1)} = c_n^{(s+2)} = \dots = c_n \quad \text{for} \quad n \leqq N_{m_{2s+1}}\,. \tag{10.11}$$

We take any polynomial $P_0(z)$ satisfying (10.9). (Conditions (10.10–11) do not restrict our choice.) E.g. $P_0(z) \equiv z$ would be an admissible choice. Suppose now that the polynomials

$$P_0(z)\ ,\ P_1(z), \dots, P_{k-1}(z)$$

and the numbers

$$m_1 < m_2 < \dots < m_{2k-2}$$

are already defined, that they satisfy condition (10.9) for $i \leqq k-1$, condition (10.10) for $s \leqq k-1$, and the condition

$$c_n^{(j)} = c_n^{(j+1)} = \dots = c_n^{(k-1)} \quad \text{for} \quad n \leqq N_{m_{2j+1}} \quad \text{and} \quad j = 1, 2, \dots, k-2 . \tag{10.12}$$

Let us write

$$\max_{|z| \leqq R_k} \left| \exp\left\{ - \sum_0^{k-1} P_i(z) \right\} \right| = A_k . \tag{10.13}$$

Since

$$e^{\sum\limits_0^{k-1} P_i(z)} = \sum_{n=0}^{\infty} c_n^{(k-1)} z^n$$

is an integral function, we have that

$$\lim_{N \to \infty} \max_{|z| \leqq R_k} \left| \sum_{n=N+1}^{\infty} c_n^{(k-1)} z^n \right| = 0 .$$

Thus we can define m_{2k-1} such that

$$m_{2k-1} > m_{2k-2} \quad \text{and} \quad \max_{|z| \leqq R_k} \left| \sum_{n=N_{m_{2k-1}}+1}^{\infty} c_n^{(k-1)} z^n \right| < \frac{M_k}{3A_k} . \tag{10.14}$$

We define now the functions: $h_k(z)$ and $Q_k(z)$ by

$$h_k(z) = \frac{\sum\limits_{n=0}^{N_{m_{2k-1}}} c_n^{(k-1)} z^n}{\sum\limits_{n=0}^{\infty} c_n^{(k-1)} z^n} = \exp\left\{ - \sum_{i=0}^{k-1} P_i(z) \right\} \cdot \sum_{n=0}^{N_{m_{2k-1}}} c_n^{(k-1)} z^n$$

$$Q_k(z) \doteq \log h_k(z)$$

($\log w$ standing for the principal value of the logarithm.)

Since

$$1 - h_k(z) = \frac{\sum\limits_{n=N_{m_{2k-1}}+1}^{\infty} c_n^{(k-1)} z^n}{\sum\limits_{n=0}^{\infty} c_n^{(k-1)} z^n} = \exp\left\{-\sum_{i=0}^{k-1} P_i(z)\right\} \cdot \sum_{n=N_{m_{2k-1}}+1}^{\infty} c_n^{(k-1)} z^n$$

we obtain, in view of (10.13) and (10.14) that

$$\max_{|z| \leqq R_k} |1 - h_k(z)| \leqq \frac{M_k}{3} \leqq \frac{1}{3}. \tag{10.15}$$

Since

$$|\log(1 - w)| \leqq |w| + \frac{|w|^2}{2} + \ldots \frac{|w|^n}{n} + \ldots = \log \frac{1}{1 - |w|} < 2\,|w|$$

$$\text{for} \quad |w| \leqq \frac{1}{3}$$

we find that, by virtue of (10.15) $Q_k(z)$ is regular in $|z| \leqq R_k$, and

$$\max_{|z| \leqq R_k} |Q_k(z)| = \max_{|z| \leqq R_k} |\log\{1 - (1 - h_k(z)\}| \leqq \frac{2}{3} M_k. \tag{10.16}$$

Since

$$Q_k(z) = \sum_{n=0}^{\infty} q_n z^n \quad \text{is regular for} \quad |z| \leqq R_k,$$

$$\lim_{N \to \infty} \max_{|z| \leqq R_k} \left| \sum_{n=N+1}^{\infty} q_n z^n \right| = 0.$$

Thus we can define m_{2k} such that

$$m_{2k} > m_{2k-1} \quad \text{and} \quad \max_{|z| \leqq R_k} \left| \sum_{n=N_{m_{2k}}+1}^{\infty} q_n z^n \right| < \frac{1}{3} M_k. \tag{10.17}$$

We define now $P_k(z)$ by

$$P_k(z) = \sum_{n=0}^{N_{m_{2k}}} q_n z^n.$$

From the inequalities (10.16) and (10.17) we obtain that, for $|z| \leqq R_k$

$$(10.18) \qquad |P_k(z)| = \left| Q_k(z) - \sum_{n=N_{m_{2k}}+1}^{\infty} q_n z^n \right| \leq |Q_k(z)| + \left| \sum_{n=N_{m_{2k}}+1}^{\infty} q_n z^n \right|$$

$$< \frac{2}{3} M_k + \frac{1}{3} M_k = M_k ;$$

(10.8) is thus satisfied for $i = k$.

Further

$$e^{P_k(z) - Q_k(z)} = \exp\left\{ - \sum_{n=N_{m_{2k}}+1}^{\infty} q_n z^n \right\} = 1 + \sum_{n=N_{m_{2k}}+1}^{\infty} b_n z^n$$

$$e^{\sum_{i=0}^{k-1} P_i(z) + Q_k(z)} = h_k(z) \exp\left\{ \sum_{i=0}^{k-1} P_i(z) \right\} = \sum_{n=0}^{N_{m_{2k-1}}} c_n^{(k-1)} z^n$$

$$e^{\sum_{i=0}^{k} P_i(z)} = \sum_{n=0}^{\infty} c_n^{(k)} z^n = \left\{ 1 + \sum_{n=N_{m_{2k}}+1}^{\infty} b_n z^n \right\} \left\{ \sum_{n=0}^{N_{m_{2k-1}}} c_n^{(k-1)} z^n \right\}.$$

From this we deduce that

$$c_n^{(k)} = c_n^{(k-1)} \qquad \text{for} \qquad n \leq N_{m_{2k-1}}$$

$$c_n^{(k)} = 0 \qquad \text{for} \qquad N_{m_{2k-1}} < n \leq N_{m_{2k}} .$$

Thus we have proved by induction the existence of a sequence $P_i(z)$ satisfying conditions (10.9—11).

Now, in view of (10.8) and (10.9), $\sum_{i=0}^{\infty} P_i(z)$ converges for every complex z. Thus

$$g(z) = \sum_{i=0}^{\infty} P_i(z)$$

is an integral function; and by virtue of (10.11)

$$f(z) = e^{g(z)} = \lim_{s \to \infty} e^{\sum_{i=0}^{s} P_i(z)} = \lim_{s \to \infty} \sum_{n=0}^{\infty} c_n^{(s)} z^n = \sum_{n=0}^{\infty} c_n z^n .$$

Finally — in view of (10.10—11) — the sequence $\{c_n\}$ obviously has the gaps, specified by (10.6).

Writing $f(z)$ in the form

$$f(z) = \sum a_n z^{\lambda_n}$$

it can be seen immediately that (10.3) will hold, whenever

$$N_{m+1} - N_m \to \infty$$

and (10.4) will hold, whenever

$$\frac{N_{m+1}}{N_m} \to \infty .$$

BIBLIOGRAPHY

[1] M. Biernacki, Sur les équations algébriques contenant des paramètres arbitraires. *Bull. de l'Acad. Polonaise des Sciences.* 1927, p. 562.

[2] J. Clunie, 'The determination of an integral function of finite order by its Taylor series'. *Journal of the London Math. Society*, 1953, p. 59—66.

[3] J. Clunie, 'On the determination of an integral function from its Taylor series', *Journal of the London Math. Society*, 1955, p. 32—42.

[4] L. Fejer, Über die Wurzel vom kleinsten absoluten Betrage einer algebraischen Gleichung'. *Math. Annalen.* 1908, p. 413.

[5] T. Kövari, 'On theorems of G. Pólya and P. Turán'. *Journal d'Analyse Mathématique.* 1958, p. 323—332.

[6] G. Pólya, 'Lücken und Singularitäten von Porenzreihen'. *Math. Zeitschrift,* 1929, p. 549—640.

[7] W. Saxer, Über die Picardschen Ausnahmewerte successiver Derivierten. *Math. Zeitschrift,* 1923, p. 206—227.

[8] P. Turán, 'Über eine neue Methode der Analysis und deren Anwendungen', Budapest, 1953.

[9] P. Turán, Über lakunäre Potenzreihen'. *Revue de Math. Pures et Appliquées,* 1956, p. 27—32.

[10] N. Wiener, 'A class of gap theorems'. *Annali di Pisa,* 1934.

[11] W. K. Hayman, 'The minimum modulus of large integral functions'. *Proc. London Math. Soc.,* 1952, p. 469—512.

[12] G. Valiron, Lectures on integral functions, 1923.

[13] A. Pfluger and G. Pólya, 'On the power series of an integral function having an exceptional value'. *Proc. Cambridge Phil. Soc.,* 1935, p. 153—155.

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(Received September 9, 1960)

THE ZEROS OF INFRAPOLYNOMIALS WITH SOME PRESCRIBED COEFFICIENTS[1]

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Introduction

The concept of *infrapolynomial*[2] [Fekete and von Neumann 1922][3] is defined as follows. Let $A(z)$ be a (complex) polynomial of the form $z^n + a_{n-1}z^{n-1} + \dots + a_0 \; (n \geqq 1)$ and let S be a compact set in the z-plane[4] containing at least n points. An *underpolynomial of* $A(z)$ *on* S is a polynomial

$$B(z) \equiv z^n + b_{n-1}z^{n-1} + \dots + b_0 \; (\not\equiv A(z)) \text{ satisfying}$$

(1) $$|B(z)| < |A(z)| \text{ whenever } z \in S \text{ and } A(z) \neq 0,$$

(2) $$B(z) = 0 \text{ whenever } z \in S \text{ and } A(z) = 0.$$

We call $A(z)$ an *infrapolynomial* on S if and only if it has no underpolynomial on S. For example, if $A(z)$ is the Tchebycheff polynomial of degree n for S (i.e., if $A(z)$ minimizes $\max\,[|P(z)|,\ z \text{ on } S]$ among all polynomials $P(z)$ of degree[5] n with leading coefficient 1), then $A(z)$ is an infrapolynomial on S.

Let n and l be integers, $0 \leqq l < n$, and let S be a compact set in the z-plane containing at least $n-l$ points. A polynomial

$$A(z) \equiv z^n + a_{n-1}z^{n-1} + \dots + a_0$$

is called an *l-fold restricted infrapolynomial on* S[6] [Walsh and Zedek 1956] if and only if there is no polynomial $B(z) \; (\not\equiv A(z))$ of

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1. This research was supported (in part) by the U.S. Air Force through the Air Force Office of Scientific Research.

2. Originally called “extremal polynomial”.

3. Dates in square brackets refer to the bibliography.

4. We deal throughout this paper with the *open* plane of complex numbers.

5. Degree of a polynomial will always mean its exact degree. The polynomial 0 is assigned the degree —1.

6. Originally called “infra-(n, l)-polynomial”.

degree n satisfying (1) and (2), whose coefficients of $z^n, z^{n-1}, \ldots, z^{n-l}$ equal the corresponding coefficients of $A(z)$.

The genesis of restricted infrapolynomials lies in the general theory of polynomial approximation to an arbitrary function on a given set [Walsh 1959, § 3]. Let $f(z)$ be a complex function defined on a finite set S in the z-plane, let $m (\geqq 0)$ be a given integer and consider the problem of minimizing $\| f(z) - P(z) \|$, where $P(z)$ ranges over all polynomials of the form $\sum_{\nu=0}^{m} c_\nu z^\nu$. Here $\| f(z) - P(z) \|$ is some classical measure of approximation. For definiteness we assume it to be

(1a) $$\max [\mu(z) | f(z) - P(z) |,\ z \text{ on } S]$$

or

(2a) $$\sum_{z \in S} \mu(z) | f(z) - P(z) |^p ,$$

where $p > 0$ and where $\mu(z)$ is a positive function.

Let $L(z)$ be Lagrange's interpolation polynomial to $f(z)$ on S. If the degree of $L(z)$ is $\leqq m$, $P(z) \equiv L(z)$ provides a solution to our problem. Moreover, in this case, a polynomial $P(z)$ is of degree $\leqq m$ and satisfies $\|f(z) - P(z)\| = 0$ if and only if $P(z) - L(z)$ is a polynomial of degree $\leqq m$ vanishing throughout S. Thus, we assume

$$L(z) \equiv \sum_{\nu=0}^{n} L_\nu z^\nu, \quad n > m, \quad L_n = 1$$

(and so, S contains more than $m+1$ points). The problem is to minimize $\| L(z) - P(z) \|$ i.e., to minimize

$$\Phi(c_0, c_1, \ldots, c_m) \equiv \left\| \sum_{\nu=0}^{m} (L_\nu - c_\nu) z^\nu + \sum_{\nu=m+1}^{n} L_\nu z^\nu \right\| .$$

The choice $L_n = 1$ involves merely an unessential normalization of $f(z)$.

Suppose we hold $f(z)$, S, and m fixed, but vary our measure of approximation over all norms (1a) and (2a), for all positive p and μ. To study simultaneously the corresponding polynomials of best approximation we introduce the concept of restricted infrapolynomials. We consider the class M of all polynomials of the form

$$\sum_{\nu=0}^{m} A_\nu z^\nu + \sum_{\nu=m+1}^{n} L_\nu z^\nu .$$

An element $A(z)$ of M may have the property that no $B(z)$ $(\in M, \not\equiv A(z))$ exists, satisfying (1) and (2). Indeed, if $P^*(z)$ is any one of the polynomials of best approximation considered, then, as is easily seen, $L(z) - P^*(z)$ $(\in M)$ is such an $A(z)$. Thus, it is appropriate to single out the elements of M with the above property. This leads to the definition of restricted infra-polynomials cited before. Each of the aforementioned polynomials $L(z) - P^*(z)$ is an $(n-m-1)$-fold restricted infra-polynomial on S. Thus the study of restricted infrapolynomials enables one to find properties shared by a whole class of polynomials of best approximation.

A similar instance giving rise to restricted infrapolynomials is the following [Walsh and Zedek, 1956]. Let

$$f(z) \equiv z^n + a_{n-1} z^{n-1} + \ldots + a_0$$

be a given polynomial, m an integer $(0 \leqq m < n)$, S a compact set in the z-plane containing at least $m+1$ points, and $\mu(z)$ a function, continuous and positive on S. If $P^*(z)$ minimizes $\max\,[\mu(z)\,|f(z) - P(z)|,\ z \text{ on } S]$ among all polynomials $P(z)$ of degree $\leqq m$, then $f(z) - P^*(z)$ is an $(n-m-1)$-fold restricted infrapolynomial on S. Here we are concerned with best approximation to a polynomial of "high" degree n by a polynomial of "low" degree $(\leqq m)$, on a compact (and perhaps infinite) set.

Conversely, every restricted infrapolynomial on a set S which vanishes nowhere on S can be written in the form $f(z) - P^*(z)$ where $P^*(z)$ minimizes a suitable norm (1a) [see Walsh 1959, § 1].

The concept of a restricted infrapolynomial can be generalized by applying (1) and (2) to polynomials

$$A(z) \equiv \sum_{\nu=0}^{n} a_\nu z^\nu \quad , \quad B(z) \equiv \sum_{\nu=0}^{n} b_\nu z^\nu$$

with $a_j = b_j$, where j ranges over some subsequence of $(0, 1, \ldots, n)$. Thus we arrive at the following definition [Shisha 1958, and for the case including $a_n = b_n = 1$, Walsh 1958].

Let n and q be natural numbers $(q \leqq n)$, $n_1, n_2, \ldots, n_q$ integers such that $0 \leqq n_1 < n_2 < \ldots < n_q \leqq n$, and S a set in the z-plane. A polynomial

$B(z)$ is called an n-th underpolynomial of a polynomial $A(z)$ on S with respect to $(n_1, n_2, \ldots, n_q)$, if and only if $A(z)$ and $B(z)$ are of degree $\leq n$

(say $A(z) \equiv \sum_{\nu=0}^{n} a_\nu z^\nu$, $B(z) \equiv \sum_{\nu=0}^{n} b_\nu z^\nu)$, $B(z) \not\equiv A(z)$, $b_{n_\nu} = a_{n_\nu}$

$(\nu = 1, 2, \ldots, q)$, and (1) and (2) hold. An n-th infrapolynomial on S with respect to $(n_1, n_2, \ldots, n_q)$ is a polynomial $A(z)$ of degree $\leq n$ having no n-th underpolynomial on S with respect to $(n_1, n_2, \ldots, n_q)$.

To illustrate, and to motivate some results of § 1, we consider the following. Let $n (\geq 2)$ be a natural number, $S (\not\ni 0)$ a finite set of $N (\geq n-1)$ points of the z-plane, and let a_0^* be a given complex number. We want to study those polynomials

$$A(z) \equiv z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0^* \tag{1b}$$

which are n-th infrapolynomials on S with respect to $(0, n)$. Such n-th infrapolynomials (on arbitrary compact sets) were studied in great detail by J. L. Walsh in a recent paper [1958].

Throughout S we have for every polynomial $A(z)$ of the form (1b),

$$A(z) = z\left[\frac{z^n + a_0^*}{z} + \sum_{\nu=1}^{n-1} a_\nu z^{\nu-1}\right] = z\left[L(z) + \sum_{\nu=0}^{n-2} a_{\nu+1} z^\nu\right]$$

where $L(z) \equiv \sum_{\nu=0}^{N-1} L_\nu z^\nu$ is Lagrange's interpolation polynomial to $\dfrac{z^n + a_0^*}{z}$ on S. If the degree δ of $L(z)$ is $\leq n-2$ then, as is easily seen, $z^n - zL(z) + a_0^*$ (which vanishes throughout S) is the unique n-th infrapolynomial on S with respect to $(0, n)$ of the form (1b). Suppose now $\delta > n-2$ (and thus $N \geq n$). For each $A(z)$ of the form (1b) we have throughout S

$$A(z) = L_\delta z\left[\sum_{\nu=0}^{n-2}\left(\frac{L_\nu + a_{\nu+1}}{L_\delta}\right) z^\nu + \sum_{\nu=n-1}^{\delta} \frac{L_\nu}{L_\delta} z^\nu\right].$$

One can show from the definitions that a polynomial (1b) is an n-th infrapolynomial on S with respect to $(0, n)$ if and only if the expression in the last square brackets is a $(\delta - n + 1)$-fold restricted infrapolynomial on S.

Consider, in particular, the case $N=n$ which implies $\delta = n-1$. We thus arrive at 0-fold restricted infrapolynomials (i.e., the infrapolynomials we mentioned in the first paragraph) of degree $n-1$ on the n-point set S. These polynomials are exactly those of the form

$$\sum_{\nu=1}^{n} \lambda_\nu \frac{g(z)}{z-z_\nu} \quad \text{where} \quad \lambda_\nu \geqq 0, \quad \sum_{\nu=1}^{n} \lambda_\nu = 1$$

[Motzkin and Walsh 1957a, Theorem 13]. Here

$$S = \{z_1, z_2, \ldots, z_n\}, \quad g(z) \equiv \prod_{\nu=1}^{n} (z-z_\nu).$$

From this it easily follows that the n-th infrapolynomials on S with respect to $(0, n)$ of the form (1b) are exactly the polynomials of the form

$$\text{(2b)} \qquad z^n + z\left[L_{n-1}\left(\sum_{\nu=1}^{n} \lambda_\nu \frac{g(z)}{z-z_\nu}\right) - L(z)\right] + a_0^*$$

where

$$\lambda_\nu \geqq 0, \quad \sum_{\nu=1}^{n} \lambda_\nu = 1.$$

By taking in particular

$$\lambda_\nu = |z_\nu g'(z_\nu)|^{-1}\left[\sum_{j=1}^{n} |z_j g'(z_j)|^{-1}\right]^{-1} \qquad (\nu = 1, 2, \ldots, n),$$

(2b) becomes the (unique) polynomial minimizing $\max [\,|A(z)|, z \text{ on } S]$ among all polynomials $A(z)$ of the form (1b). This can be proved by a reduction similar to the one just made, using a formula of Motzkin and Walsh [1953] for the Tchebycheff polynomial of degree $n-1$ for an n-point set.

We return to a more general situation.

For every natural number n, we define a "simple n-sequence" to be a sequence having one of the forms $(0, 1, \ldots, k, n-l, n-l+1, \ldots, n)$ $[k \geqq 0;\ l \geqq 0,\ k+l+2 \leqq n]$, $(0, 1, \ldots, k)$ $[0 \leqq k < n]$, $(n-l, n-l+1, \ldots, n)$ $[0 \leqq l < n]$. Thus the complement with respect to $(0, 1, \ldots, n)$ of a simple n-sequence is a sequence without "gaps".

In the present paper we deal with n-th infrapolynomials with respect to simple n-sequences, and our main aim is to obtain information on the geometric location of the zeros of these polynomials.

It is to be observed that every n-th infrapolynomial on a set S with respect to a sequence $(n_1, n_2, \ldots, n_q)$ is also an n-th infrapolynomial on S with respect to a simple n-sequence σ. Indeed, as such a σ one can take any simple n-sequence containing $(n_1, n_2, \ldots, n_q)$ as a subsequence.

In Theorem 1 we show that given a simple n-sequence σ of s elements and a finite set S ($\not\ni 0$ if $0 \in \sigma$) of N elements, the problem of locating the zeros of the n-th infrapolynomials on S with respect to σ is essentially trivial whenever $N < n - s + 2$. Furthermore, for the first non-trivial case $N = n - s + 2$, we determine explicitly these n-th infrapolynomials (Theorem 1, c).

We get similar, but more precise information from Theorem 2. We prove this theorem by the method we have illustrated above for the case $\sigma = (0, n)$.

With the help of Theorems 1 and 2 we are able to obtain various results on the location of the zeros of n-th infrapolynomials on sets of $n - s + 2$ points, with respect to simple n-sequences σ [Theorems 6, 7, 8, 14 and 15].

In Theorems 3 and 4 we show that such n-th infrapolynomials are closely related in their structure to certain combinations of a polynomial and its derivative. This generalizes an observation of Fekete and von Neumann [1922] as elaborated by Motzkin and Walsh [1953].

There is a further reason for emphasizing n-th infrapolynomials on sets of $n - s + 2$ points. Let $A(z)$ ($\not\equiv 0$) be an n-th infrapolynomial on a compact S, with respect to some simple n-sequence σ of s elements ($0 \notin S$ if $0 \in \sigma$). We set $A(z) \equiv B(z)\, D(z)$, where $D(z)$ is a polynomial all of whose zeros belong to S, whereas $B(z)$ is a polynomial vanishing nowhere on S. Assume that the degree r of $B(z)$ is $\geqq s$. By an unpublished result of O. Shisha generalizing a theorem of M. Fekete [1951; Theorem I, cf. also II, 12] $B(z)$ is a divisor of some $Q(z)$, an m-th ($r \leqq m \leqq 2r - s + 1$) infrapolynomial on an $(m - s + 2)$-point subset S_0 of S, with respect to some simple m-sequence σ_1 ($\not\ni 0$ if $0 \in S_0$) having s elements. By studying the location of the zeros of $Q(z)$ with respect to S_0 one may obtain results

on the location of the zeros of $A(z)$ with respect to S. We prove a number of theorems by this method.

After discussing in § 1 general n-th infrapolynomials on some finite sets (with emphasis on $(n-s+2)$-point sets), we devote § 2 to n-th infrapolynomials with respect to $(0, n)$. In § 3 we treat n-th infrapolynomials with respect to $(n-1, n)$.

Very important tools in §§ 2, 3 are theorems of J. L. Walsh on the distribution of the roots of certain equations [1922, Theorem VI, 1958 Corollary 1 to Theorem 8]. We continue in the latter direction in § 4, and obtain results on the geometric location of the zeros of polynomials and other rational functions, closely related in their structure to the n-th infrapolynomials discussed in §§ 1—3.

§ **1.** The n-th infrapolynomials on some finite sets with respect to simple n-sequences.

Theorem 1.

Hypotheses:

1. n is an arbitrary natural number, σ a simple n-sequence, s its number of elements.
2. $S(\neq \emptyset)$[7] is a finite set in the z-plane, N its number of points,

$$g(z) \equiv \prod_{v=1}^{N} (z - z_v)$$

where $S = \{z_1, \ldots, z_N\}$.

3. In case $\sigma = (0, 1, \ldots, k)$ or $\sigma = (0, 1, \ldots, k, n-l, n-l+1, \ldots, n)$ we set $K = k+1$. In case $0 \notin \sigma$, we set $K = 0$. Thus $K = \min [\nu, \nu \notin \sigma, \nu = 0, 1, 2, \ldots]$.
4. In case $0 \in \sigma$, we assume $0 \notin S$.[8]

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7. There exists no n-th infrapolynomial on the empty set $\emptyset$ with respect to σ. For let

$$A(z) \equiv \sum_{\nu=0}^{n} a_\nu z^\nu$$

be any polynomial of degree $\leq n$. Let μ be an integer such that $0 \leq \mu \leq n$, $\mu \notin \sigma$. Then

$$B(z) \equiv (a_\mu + 1) z^\mu + \sum_{\nu \in \sigma} a_\nu z^\nu$$

is an n-th underpolynomial of $A(z)$ on $\emptyset$ with respect to σ.

8. We shall discuss at the end of this section the case $0 \in \sigma$, $0 \in S$.

Conclusions:

a. If $N \leqq n-s$, there exists no n-th infrapolynomial on S with respect to σ.

b. If $N=n-s+1$, a polynomial $A(z)$ of degree $\leqq n$ is an n-th infrapolynomial on S with respect to σ, if and only if it vanishes throughout S.

c. If $N=n-s+2$, a polynomial $A(z)$ is an n-th infrapolynomial on S with respect to σ if and only if it is of the form

$$P(z)\,g(z)+\alpha z^{K}\sum_{\nu=1}^{N}\lambda_\nu\,\frac{g(z)}{z-z_\nu} \tag{3}$$

where $\lambda_1, \ldots, \lambda_N$ are non-negative reals with

$$\sum_{\nu=1}^{N}\lambda_\nu = 1,$$

α is a complex number, and $P(z)$ a polynomial of degree $\leqq s-1$ such that $P(z)\,g(z)+\alpha z^{K+N-1}$ is of degree $\leqq n$.[9]

It may be noted that there are N unprescribed coefficients of a possible n-th underpolynomial of $A(z)$ in case b, $N-1$ in case c.

We defer the proof until after that of the next theorem.

Theorem 2.

Hypotheses:

1. The hypotheses of Theorem 1.
2. a_ν^* is a given complex number for each $\nu \in \sigma$. Π denotes the set of all polynomials of degree $\leqq n$, whose coefficient of each z^ν ($\nu \in \sigma$) is a_ν^*.
3. $L(z) \equiv L_{N-1}z^{N-1}+\ldots+L_0$ is Lagrange's interpolation polynomial to

$$z^{-K}\sum_{\nu\in\sigma} a_\nu^* z^\nu \quad \text{on } S, \text{ and } \quad A_0(z) \equiv \Big(\sum_{\nu\in\sigma} a_\nu^* z^\nu\Big) - z^K L(z).$$

9. In case σ is of one of the forms $(0, 1, \ldots, k, n-l, n-l+1, \ldots, n)$, $(n-l, n-l+1, \ldots, n)$, the conditions on $P(z)$ can be formulated more simply: $P(z)$ is a polynomial of degree $\leqq s-2$. In case σ is of the form $(0, 1, \ldots, k)$, they can be formulated: $P(z)$ is a polynomial of degree $\leqq s-1$, whose coefficient of z^{s-1} is $-\alpha$.

Conclusions:

a. If $N = n - s + 1$, $A_0(z)$ is the unique n-th infrapolynomial on S with respect to σ, belonging to Π.

b. If $N = n - s + 2$, a polynomial $A(z)$ is an n-th infrapolynomial on S, with respect to σ, belonging to Π, if and only if it is of the form

$$A_0(z) + L_{N-1} z^K \sum_{\nu=1}^{N} \lambda_\nu \frac{g(z)}{z - z_\nu} \tag{4}$$

where $\lambda_\nu \geqq 0$, $\sum_{\nu=1}^{N} \lambda_\nu = 1$.

Remark: Observe that $A_0(z)$ vanishes throughout S. Thus in case $N = n - s + 2$, we may set

$$A_0(z) \equiv \left(\sum_{\nu=0}^{s-1} \alpha_\nu^* z^\nu\right) g(z). \tag{5}$$

In particular, if σ is of the form $(0, 1, \ldots, k, n)$ or $(0, 1, \ldots, k)$, we have

$$A_0(z) \equiv \left(\sum_{\nu \in \sigma} a_\nu^* z^\nu\right) - z^{k+1} L(z) \equiv \left(\sum_{\nu=0}^{k} \alpha_\nu^* z^\nu\right) g(z). \tag{6}$$

For $j = 0, 1, \ldots, k$, we get from (6) by Leibniz's rule for differentiating a product

$$\left[\frac{d^j}{dz^j}\left(\frac{1}{g(z)} \sum_{\nu \in \sigma} a_\nu^* z^\nu\right)\right]_{z=0} = \left[\frac{d^j}{dz^j} \sum_{\nu=0}^{k} \alpha_\nu^* z^\nu\right]_{z=0} = j!\, \alpha_j^*,$$

$$\alpha_j^* = \sum_{\nu=0}^{j} \frac{1}{\nu!} \left(\frac{1}{g(z)}\right)_{z=0}^{(\nu)} a_{j-\nu}^*. \tag{7}$$

From (6) we observe also that if $\sigma = (0, 1, \ldots, k, n)$, then $L_{N-1} = a_n^* - \alpha_k^*$, so that (4) can be put in the form

$$\left(\sum_{\nu=0}^{k} \alpha_\nu^* z^\nu\right) g(z) + (a_n^* - \alpha_k^*)\, z^{k+1} \sum_{\nu=1}^{N} \lambda_\nu \frac{g(z)}{z - z_\nu}. \tag{8}$$

If $\sigma = (0, 1, \ldots, k)$, we have $L_{N-1} = -\alpha_k^*$.

Proof of Theorem 2: We consider first the two cases (i) $N = n - s + 1$;

and (ii) $N=n-s+2$, $L_{N-1}=0$. If $B(z)$ is a polynomial belonging to Π and vanishing throughout S, we can set

$$B(z) \equiv \left(\sum_{\nu\in\sigma} a_\nu^* z^\nu\right) - z^K l(z),$$

where $l(z)$ is a polynomial of degree $\leq n-s$ which equals $z^{-K}\sum_{\nu\in\sigma} a_\nu^* z^\nu$ on S. Thus $l(z)\equiv L(z)$, and therefore $B(z)\equiv A_0(z)$. Hence $A_0(z)$ is the unique polynomial belonging to Π and vanishing throughout S. Thus, $A_0(z)$ has no n-th underpolynomial on S with respect to σ, but is itself such an n-th underpolynomial of every $A(z)\,(\in\Pi, \not\equiv A_0(z))$. Hence the desired conclusion.

Let now $N=n-s+2$, $L_{N-1}\neq 0$. Let $\tilde{\Pi}$ denote the set of all polynomials of degree $n-s+1$ with leading coefficient 1. For every $A(z)\in\Pi$, let $\tilde{A}(z)$ be the element of $\tilde{\Pi}$ satisfying

$$A(z) \equiv \left(\sum_{\nu\in\sigma} a_\nu^* z^\nu\right) + z^K[-L(z)+L_{N-1}\tilde{A}(z)]. \tag{9}$$

We observe that for every $B_1(z)\in\tilde{\Pi}$,

$$B(z) \equiv \left(\sum_{\nu\in\sigma} a_\nu^* z^\nu\right) + z^K[-L(z)+L_{N-1}B_1(z)]$$

belongs to Π, and $B_1(z)\equiv\tilde{B}(z)$.

Equation (9) implies

$$A(z) = L_{N-1}z^K\tilde{A}(z) \qquad \text{throughout } S. \tag{10}$$

From (9) and (10) it follows that if $A(z), B(z)\in\Pi$, $B(z)$ is an n-th underpolynomial of $A(z)$ on S with respect to σ, if, and only if, $\tilde{B}(z)$ is an underpolynomial of $\tilde{A}(z)$ on S. Thus, an element $A(z)$ of Π is an n-th infrapolynomial on S with respect to σ, if and only if $\tilde{A}(z)$ is an infrapolynomial on S, i.e. [Motzkin and Walsh 1957a, Theorem 13] if, and only if, $\tilde{A}(z)$ is of the form

$$\sum_{\nu=1}^{N}\lambda_\nu\frac{g(z)}{z-z_\nu}\left(\lambda_\nu\geq 0,\quad\sum_{\nu=1}^{N}\lambda_\nu=1\right).$$

From this (observing that every polynomial of the form (4) belongs to Π) the result follows.

Theorem 2 includes the case $N=n-s+2$, $s=1$, $\sigma=(n)$, $a_n^*=1$, $K=0$, whence $L(z)\equiv L_{N-1}z^n+\ldots\equiv z^n$, $A_0(z)\equiv 0$; thus an $A(z)$ of Π is an n-th infrapolynomial on S with respect to σ if, and only if,

$$A(z)\equiv \tilde{A}(z)\equiv \sum_{\nu=1}^{N}\lambda_\nu \frac{g(z)}{z-z_\nu},$$

which is the theorem of Motzkin and Walsh just cited.

Proof of Theorem 1: Let

$$A(z)\equiv \sum_{\nu=0}^{n} a_\nu^* z^\nu$$

be an arbitrary polynomial of degree $\leqq n$, and let Π, $L(z)$, L_{N-1} and $A_0(z)$ be as in Theorem 2.

Suppose $N\leqq n-s$.

We define $B(z)$ as follows: it is $A_0(z)$ if $A_0(z)\not\equiv A(z)$, and otherwise it is $A_0(z)+z^K g(z)$. Then $B(z)$ $[\not\equiv A(z)]$ vanishes throughout S, and belongs to Π. Thus, $B(z)$ is an n-th underpolynomial of $A(z)$ on S with respect to σ, which proves (a).

Let now $N=n-s+1$. We saw in the proof of Theorem 2 that $A_0(z)$ is the unique element of Π vanishing throughout S. By conclusion (a) of Theorem 2, $A(z)$ is an n-th infrapolynomial on S with respect to σ, if, and only if, $A(z)\equiv A_0(z)$, i.e., if, and only if, $A(z)=0$ throughout S.

Consider finally the case $N=n-s+2$. If $A(z)$ is an n-th infrapolynomial on S with respect to σ, then by Theorem 2 it is of the form (4), and therefore, by (5), it is of the form (3). Conversely, let $A(z)$ be of the form (3), so that

$$A(z)\equiv \sum_{\nu=0}^{n} a_\nu^* z^\nu \equiv P(z)g(z)+\alpha z^{K+N-1}+\alpha z^K\left(-z^{N-1}+\sum_{\nu=1}^{N}\lambda_\nu\frac{g(z)}{z-z_\nu}\right).$$

Consider the polynomial

$$Q(z)\equiv \alpha z^K\left(-z^{N-1}+\sum_{\nu=1}^{N}\lambda_\nu\frac{g(z)}{z-z_\nu}\right).$$

It is of degree $\leqq n$, and its coefficient of z^ν is 0, for every $\nu\in\sigma$. Therefore,

$$R(z)\equiv P(z)\,g(z)+\alpha z^{K+N-1}$$

is also of degree $\leqq n$, and we can write

$$R(z) \equiv \left(\sum_{\nu \in \sigma} a_\nu^* z^\nu\right) - z^K M(z),$$

where $M(z)$ is a polynomial of degree $\leqq n - s \, (= N - 2)$. Throughout S we have

$$z^K [M(z) + \alpha z^{N-1}] = \left(\sum_{\nu \in \sigma} a_\nu^* z^\nu\right) - R(z) + \alpha z^{K+N-1} = \sum_{\nu \in \sigma} a_\nu^* z^\nu,$$

and therefore by the definitions of $L(z)$, L_{N-1} and $A_0(z)$ we have

$$M(z) + \alpha z^{N-1} \equiv L(z), \; \alpha = L_{N-1}.$$

Thus,

$$A(z) \equiv \left(\sum_{\nu \in \sigma} a_\nu^* z^\nu\right) - z^K [L(z) - \alpha z^{N-1}] + Q(z) \equiv A_0(z) + L_{N-1} z^K \sum_{\nu=1}^{N} \lambda_\nu \frac{g(z)}{z - z_\nu}.$$

By Theorem 2, $A(z)$ is an n-th infrapolynomial on S with respect to σ. This completes the proof.

Theorem 3: Let the hypotheses of Theorem 1 hold, with $N = n - s + 2$. For every complex number β, and every polynomial $P(z)$ of degree $\leqq s-1$, $P(z) g(z) + \beta z^K g'(z)$, if of degree $\leqq n$, is an n-th infrapolynomial on S with respect to σ. More generally: let $G(z)$ be a non-constant polynomial all of whose zeros (not necessarily simple) belong to S, β an arbitrary complex number, and $P(z)$ an arbitrary polynomial of degree $\leqq s-1$. Then

$$A(z) \equiv P(z) g(z) + \beta z^K \frac{G'(z) \, g(z)}{G(z)},$$

if of degree $\leqq n$, is an n-th infrapolynomial on S with respect to σ.

Proof: For $\nu = 1, 2, \ldots, N$, let $m_\nu (\geqq 0)$ be the multiplicity of z_ν as a zero of $G(z)$, and let $\lambda_\nu = \dfrac{m_\nu}{M}$, where $M = \sum_{j=1}^{N} m_j$. Then

$$G'(z) \equiv \sum_{G(z_\nu)=0} m_\nu \frac{G(z)}{z-z_\nu},$$

so that

$$A(z) \equiv P(z)\, g(z) + \beta M z^K \sum_{\nu=1}^{N} \lambda_\nu \frac{g(z)}{z-z_\nu},$$

and we can apply Theorem 1.

Remark: The *polar derivative* of a non-constant polynomial $f(z)$ with respect to a point z_0 of the z-plane is defined as

$$mf(z) - (z - z_0) f'(z),$$

where m is the degree of $f(z)$ [Laguerre 1898. We use the terminology of Marden 1949.] From Theorem 3 we get: if $f(z)$ is a polynomial whose degree m is $\geqq 2$, and whose zeros are simple and $\neq 0$, then its polar derivative with respect to 0 is an $(m-1)$-th infrapolynomial on the set of zeros of $f(z)$ with respect to (0). More generally: let $f(z)$ be a polynomial of degree $\geqq 2$, and $z_1, z_2, \ldots, z_N$ its distinct zeros, and suppose $N \geqq 2$, $0 \notin S = \{z_1, \ldots, z_N\}$. Then for every complex number β, $\beta \dfrac{f_1(z)\, g(z)}{f(z)}$ is an $(N-1)$-th infrapolynomial on S with respect to (0). Here

$$g(z) \equiv \prod_{\nu=1}^{N} (z - z_\nu),$$

and $f_1(z)$ is the polar derivative of $f(z)$ with respect to 0.

The following is, in a limiting sense, a converse of Theorem 3.

Theorem 4: *Let the hypotheses of Theorem 1 hold, let $N = n - s + 2$, and let $A(z)$ be an n-th infrapolynomial on S with respect to σ. Then there exists a sequence $(G_j(z))_{j=1}^{\infty}$ of non-constant polynomials all of whose zeros belong to S, a sequence $(\beta_j)_{j=1}^{\infty}$ of complex numbers, and a polynomial $P(z)$ of degree $\leqq s-1$, such that*

$$A_j(z) \equiv P(z)\, g(z) + \beta_j z^K \frac{G'_j(z)\, g(z)}{G_j(z)}$$

converges uniformly to $A(z)$ on every bounded set.

Proof: Write $A(z)$ in the form (3). For $\nu = 1, 2, \ldots, N-1$, let $(\lambda_\nu^{(j)})_{j=1}^{\infty}$ be a sequence of non-negative rationals $\leqq \lambda_\nu$, converging to λ_ν. Let for $j = 1, 2, \ldots$,

$$\lambda_N^{(j)} = 1 - \sum_{\nu=1}^{N-1} \lambda_\nu^{(j)} \quad \left(\geqq 1 - \sum_{\nu=1}^{N-1} \lambda_\nu = \lambda_N \geqq 0\right).$$

Then also $\lim_{j\to\infty} \lambda_N^{(j)} = \lambda_N$.

Set $\lambda_\nu^{(j)} = \dfrac{m_\nu^{(j)}}{d_j}$, $\beta_j = \dfrac{\alpha}{d_j}$ $(\nu = 1, 2, \ldots, N, \quad j = 1, 2, \ldots)$ where $m_\nu^{(j)}$ $(\geqq 0)$ and $d_j\,(>0)$ are integers. Finally, for $j = 1, 2, \ldots,$ define

$$G_j(z) \equiv \prod_{\nu=1}^{N} (z - z_\nu)^{m_\nu^{(j)}}.$$

Then

$$A_j(z) \equiv P(z)g(z) + \beta_j z^K \frac{G'_j(z)\, g(z)}{G_j(z)} \equiv P(z)g(z) + \alpha z^K \sum_{\nu=1}^{N} \lambda_\nu^{(j)} \frac{g(z)}{z - z_\nu}$$

converges uniformly to $A(z)$ on every bounded set. Observe that by Theorem 1, each $A_j(z)$ is itself an n-th infrapolynomial on S with respect to σ.

Theorem 5: Let the hypotheses of Theorem 2 hold, and let $\mu(z)$ be a function, defined and positive on S.

a. If $N \leqq n - s$, there exist infinitely many polynomials $T(z)$, belonging to Π and satisfying

(11) $$\max\,[\mu(z)\,|T(z)|,\ z \text{ on } S] \leqq \max\,[\mu(z)\,|A(z)|,\ z \text{ on } S]$$

for every $A(z) \in \Pi$.

b. If $N = n - s + 1$, $A_0(z)$ is the only such $T(z)$.

c. If $N = n - s + 2$,

$$T_0(z) \equiv A_0(z) + L_{N-1} z^K \sum_{\nu=1}^{N} \lambda_\nu^* \frac{g(z)}{z - z_\nu}$$

is the only such $T(z)$. Here

$$\lambda_\nu^* = |\mu(z_\nu)\, z_\nu^K g'(z_\nu)|^{-1} \left[\sum_{j=1}^{N} |\mu(z_j)\, z_j^K g'(z_j)|^{-1}\right]^{-1} \quad (\nu = 1, 2, \ldots, N).$$

Proof: If $N \leqq n - s$, then for each number α,

$$T_\alpha(z) \equiv A_0(z) + \alpha z^K g(z) \equiv \left(\sum_{\nu \in \sigma} a_\nu^* z^\nu\right) + z^K[-L(z) + \alpha g(z)]$$

belongs to Π and vanishes throughout S. This proves (a).

Consider next the two cases: $N = n - s + 1$; and $N = n - s + 2$, $L_{N-1} = 0$. $A_0(z)$ is, as was shown in the proof of Theorem 2, the unique element of Π vanishing throughout S. From this follows the result.

Suppose finally, $N = n - s + 2$, $L_{N-1} \neq 0$. We use notations of the proof of Theorem 2. Equation (10) implies that for every $A(z) \in \Pi$,

(11a) $\max[\mu(z)|A(z)|, z \text{ on } S] = |L_{N-1}| \max[\mu(z)|z^K \tilde{A}(z)|, z \text{ on } S]$.

As mentioned in that proof, every $B_1(z) \in \tilde{\Pi}$ is a $\tilde{B}(z)$ for some $B(z) \in \Pi$.

From (11a) it follows that a $T(z)$ belonging to Π satisfies (11), if, and only if,

$$\max[\mu(z)|z^K \tilde{T}(z)|, z \text{ on } S] \leqq \max[\mu(z)|z^K B_1(z)|, z \text{ on } S]$$

for every $B_1(z) \in \tilde{\Pi}$, i.e., [Motzkin and Walsh, 1953] if, and only if,

$$\tilde{T}(z) \equiv \sum_{\nu=1}^{N} \lambda_\nu^* \frac{g(z)}{z - z_\nu},$$

that is, if, and only if, $T(z) \equiv T_0(z)$. Since $T_0(z) \in \Pi$, this completes the proof.

Theorem 5 determines explicitly, in the cases $N = n - s + 1$ and $N = n - s + 2$, the polynomials $T(z)$ of least Tchebycheff norm (1a) on S with weight function $\mu(z)$ among all polynomials of Π. Theorem 5 extends readily to include the polynomials of Π minimizing the norm (2a); see [Motzkin and Walsh, 1955, 1956].

We now illustrate our results by an example. Let $z_1, \dots, z_n$ $(n \geqq 2)$ be distinct points of the z-plane, set

$$g(z) \equiv \prod_{\nu=1}^{n} (z - z_\nu),$$

and assume $0 \notin S = \{z_1 . z_2, \dots, z_n\}$. By Theorem 1, a polynomial $A(z)$ is an n-th infrapolynomial on S with respect to $(0, n)$, if, and only if,

it is of the form

$$\alpha_0 g(z) + \alpha z \sum_{\nu=1}^{n} \lambda_\nu \frac{g(z)}{z-z_\nu},$$

where α_0 and α are complex numbers, and λ_ν are non-negative reals with $\sum_{\nu=1}^{n} \lambda_\nu = 1$. In particular (cf. Theorem 3), for every pair α_0, β of complex numbers, $\alpha_0 g(z) + \beta z g'(z)$ is an n-th infrapolynomial on S with respect to $(0, n)$. Let a_0^* and a_n^* be any given complex numbers. By Theorem 2 and the remark following it, a polynomial $A(z)$ is an n-th infrapolynomial on S with respect to $(0, n)$ having a_0^* and a_n^* as its coefficients of z^0 and z^n respectively, if, and only if, it is of the form (cf. (8) and (7))

$$\frac{a_0^*}{g(0)} g(z) + \left(a_n^* - \frac{a_0^*}{g(0)}\right) z \sum_{\nu=1}^{n} \lambda_\nu \frac{g(z)}{z-z_\nu} \tag{12}$$

where $\lambda_\nu \geqq 0$, $\sum_{\nu=1}^{n} \lambda_\nu = 1$.

We turn now to geometric properties of the zeros of infrapolynomials with some prescribed coefficients. Given a set E in the z-plane, we denote by E^* its convex hull.

Theorem 6: *Let the hypotheses of Theorem 2 hold, with $N = n-s+2$. Denote by Π_1 the set of all n-th infrapolynomials on S with respect to σ, belonging to Π.*

a. *If σ is of the form $(0, 1, \ldots, k, n-l, n-l+1, \ldots, n)$ or $(n-l, n-l+1, \ldots, n)$, if $l > 0$, and if*

$$\sum_{\nu=n-l+1}^{n} |a_\nu^*| > 0,$$

then there exists a positive H, such that every zero of every $A(z) \in \Pi_1$, is of modulus $\leqq H$.

b. *If σ is of the form $(0, 1, \ldots, k, n-l, n-l+1, \ldots, n)$ or $(0, 1, \ldots, k)$, if*

$$\sum_{\nu=0}^{k} |a_\nu^*| > 0,$$

and if $0 \notin S^*$, then there exists a positive h such that every zero ($\neq 0$) of every $A(z) \in \Pi_1$ is of modulus $\geq h$.

We shall use in the proof (as well as in later sections) the following

Lemma: Let $z_0, z_1, \ldots, z_m$ ($m \geq 1$, $z_0 \neq z_\nu$ for $\nu = 1, 2, \ldots, m$) be points of the z-plane, and let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be non-negative reals with

$$\sum_{\nu=1}^{m} \lambda_\nu = 1.$$

Then there exists a c^* ($\neq z_0$) belonging to $\{z_1, z_2, \ldots, z_m\}^*$, and a λ, $0 \leq \lambda \leq 1$, such that

$$\sum_{\nu=1}^{m} \frac{\lambda_\nu}{z_0 - z_\nu} = \frac{\lambda}{z_0 - c^*}.$$

Proof of the Lemma: Set

$$a = \sum_{\nu=1}^{m} \frac{\lambda_\nu}{z_0 - z_\nu}. \tag{13}$$

If $a = 0$, we may take $c^* = z_1$, $\lambda = 0$. We suppose therefore that $a \neq 0$. From (13) we have

$$\bar{a} = z_0 \sum_{\nu=1}^{m} \lambda_\nu |z_0 - z_\nu|^{-2} - \sum_{\nu=1}^{m} \lambda_\nu z_\nu |z_0 - z_\nu|^{-2}.$$

Let

$$l_\nu = \lambda_\nu |z_0 - z_\nu|^{-2} \left[\sum_{j=1}^{m} \lambda_j |z_0 - z_j|^{-2}\right]^{-1} \quad (\nu = 1, 2, \ldots, m),$$

$$c^* = \sum_{\nu=1}^{m} l_\nu z_\nu.$$

Then

$$c^* \in \{z_1, z_2, \ldots, z_m\}^*,$$

and

$$\bar{a} \left[\sum_{\nu=1}^{m} \lambda_\nu |z_0 - z_\nu|^{-2}\right]^{-1} = z_0 - c^* \qquad (\neq 0).$$

Thus

$$a = |a|^2 \frac{1}{z_0 - c^*} \left[\sum_{\nu=1}^{m} \lambda_\nu |z_0 - z_\nu|^{-2} \right]^{-1}. \tag{14}$$

From (13) we get

$$|a|^2 \leq \left(\sum_{\nu=1}^{m} \frac{\lambda_\nu}{|z_0 - z_\nu|} \right)^2 = \left(\sum_{\nu=1}^{m} \sqrt{\lambda_\nu} \frac{\sqrt{\lambda_\nu}}{|z_0 - z_\nu|} \right)^2 \leq \sum_{\nu=1}^{m} \frac{\lambda_\nu}{|z_0 - z_\nu|^2}.$$

Setting

$$\lambda = |a|^2 \left[\sum_{\nu=1}^{m} \lambda_\nu |z_0 - z_\nu|^{-2} \right]^{-1},$$

we have $0 < \lambda \leqq 1$. Also, by (13) and (14),

$$\sum_{\nu=1}^{m} \frac{\lambda_\nu}{z_0 - z_\nu} = a = \frac{\lambda}{z_0 - c^*}.$$

Proof of Theorem 6: Let the hypotheses of part (a) of the Theorem hold. Then $K \leqq s - 2$. Also, since the degree of $A_0(z)$ is $> n - l$, (5) implies that

$$\sum_{\nu=K}^{s-1} |\alpha_\nu^*| > 0.$$

We choose a positive H ($\geqq \max [|z|, z \text{ on } S]$) such that

$$|(z - c) z^{-K} \sum_{\nu=0}^{s-1} \alpha_\nu^* z^\nu| > |L_{N-1}| \tag{15}$$

whenever $|z| > H$ and $c \in S^*$.

Let z_0 be a zero of an $A(z) \in \Pi_1$. Suppose first that $z_0 \notin S$. By Theorem 2 and by (5), an equality

$$\left(\sum_{\nu=0}^{s-1} \alpha_\nu^* z_0^\nu \right) + L_{N-1} z_0^K \sum_{\nu=1}^{N} \frac{\lambda_\nu}{z_0 - z_\nu} = 0$$

holds for some $\lambda_\nu (\geqq 0)$ with

$$\sum_{\nu=1}^{N} \lambda_\nu = 1.$$

By the Lemma, for some $c^* \in S^*$ and some λ $(0 \leqq \lambda \leqq 1)$ we have

(16) $$\left(\sum_{\nu=0}^{s-1} \alpha_\nu^* z_0^\nu\right) + L_{N-1} z_0^K \frac{\lambda}{z_0 - c^*} = 0.$$

Therefore, by (15), $|z_0| \leqq H$. If $z_0 \in S$, then $|z_0| \leqq H$ by the definition of H.

Suppose now the hypotheses of part (b) hold. Then

$$\sum_{\nu=0}^{k} |\alpha_\nu^*| > 0.$$

Choose an h $(0 < h < \min [|z|, z \text{ on } S])$ such that

(17) $$|(z-c) z^{-K} \sum_{\nu=0}^{s-1} \alpha_\nu^* z^\nu| > |L_{N-1}|$$

whenever $0 < |z| < h$ and $c \in S^*$.

Let $z_0 (\neq 0)$ be a zero of an $A(z) \in \Pi_1$. If $z_0 \notin S$ we have again a relation (16) with $c^* \in S^*$, $0 \leqq \lambda \leqq 1$, and by (17), $|z_0| \geqq h$. If $z_0 \in S$, then $|z_0| \geqq h$ by the definition of h.

We conclude § 1 with the following remarks [10], concerning especially sets S containing 0 and sequences (n_j) containing 0.

I. Let $n, n_1, n_2, \ldots, n_q$ $(1 \leqq q \leqq n,\ 0 = n_1 < n_2 < \ldots < n_q \leqq n)$ be integers, and S a set in the z-plane

a. If $0 \in S$, every polynomial $\sum_{\nu=0}^{n} a_\nu z^\nu$ with $a_0 \neq 0$ is an n-th infrapolynomial on S with respect to $(n_1, n_2, \ldots, n_q)$.

b. If $q \geqq 2$, a polynomial $\sum_{\nu=1}^{n} a_\nu z^\nu$ is an n-th infrapolynomial on S with respect to $(n_1, \ldots, n_q)$, if, and only if, $\sum_{\nu=1}^{n} a_\nu z^{\nu-1}$ is an $(n-1)$-th infrapolynomial on $S - \{0\}$ with respect to $(n_2 - 1, \ldots, n_q - 1)$.

To prove (a), we observe that if $B(z)$ is an n-th underpolynomial of a polynomial $A(z)$ with $A(0) \neq 0$, on $S (\ni 0)$, with respect to $(n_1, n_2, \ldots, n_q)$, then we have both $B(0) = A(0)$, $|B(0)| < |A(0)|$.

(b) is easily proved from the definition of an infrapolynomial with respect to a sequence.

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10. Compare with [Walsh 1958], last paragraph on p. 297 and the one following.

II. Let n be a natural number and S a set in the z-plane.

a. Suppose that either (i) S contains fewer than n points, or (ii) S contains exactly n points one of which is 0. Then there exists no n-th infrapolynomial on S with respect to (0) of the form $\sum_{\nu=1}^{n} a_\nu z^\nu$.

b. Suppose S (finite or infinite) contains more than n points. Then the polynomial 0 is the unique n-th infrapolynomial on S with respect to (0) of the form $\sum_{\nu=1}^{n} a_\nu z^\nu$.

Proof. In case (a), given any polynomial

$$A(z) \equiv \sum_{\nu=1}^{n} a_\nu z^\nu,$$

we can find a polynomial $B(z)$ $(\not\equiv A(z))$ of degree n, vanishing throughout $S \cup \{0\}$. This $B(z)$ is an n-th underpolynomial of $A(z)$ on S with respect to (0). In case (b), 0 is obviously an n-th underpolynomial of every polynomial

$$\sum_{\nu=1}^{n} a_\nu z^\nu \ (\not\equiv 0)$$

on S with respect to (0), but 0 has no n-th underpolynomial on S with respect to (0).

III. In Theorem 1 we assumed that if $0 \in \sigma$, $0 \notin S$. Suppose now that hypotheses 1—3 of Theorem 1 hold, and that $0 \in \sigma$, $0 \in S$. We want, as in Theorem 1, to determine all n-th infrapolynomials on S with respect to σ, whenever N is smaller than some bound. Because of Ia, we can restrict ourselves to polynomials of the form $\sum_{\nu=1}^{n} a_\nu z^\nu$, and because of IIa and IIb we may assume $\sigma \neq (0)$. Let

$$\sigma = (0,\ n_2,\ n_3,\ \ldots,\ n_s).$$

Then

$$\sigma_1 = (n_2 - 1,\ n_3 - 1,\ \ldots,\ n_s - 1)$$

is a simple $(n-1)$-sequence, and by Ib, a polynomial $\sum_{\nu=1}^{n} a_\nu z^\nu$ is an n-th

infrapolynomial on S with respect to σ, if, and only if, $\sum_{\nu=1}^{n} a_\nu z^{\nu-1}$ is an $(n-1)$-th infrapolynomial on $S-\{0\}$ with respect to σ_1. By applying Theorem 1 to the polynomials $\sum_{\nu=1}^{n} a_\nu z^{\nu-1}$ one easily shows:

a. If $N \leqq n-s+1$, there exists no n-th infrapolynomial on S with respect to σ of the form $\sum_{\nu=1}^{n} a_\nu z^\nu$.

b. If $N = n-s+2$, a polynomial $\sum_{\nu=1}^{n} a_\nu z^\nu$ is an n-th infrapolynomial on S with respect to σ if, and only if, it vanishes throughout S.

c. If $N = n-s+3$, a polynomial $\sum_{\nu=1}^{n} a_\nu z^\nu$ is an n-th infrapolynomial on S with respect to σ if, and only if, it is of the form

$$P(z)\,g(z) + \alpha z^{K-1} \sum_{\substack{\nu=1 \\ z_\nu \neq 0}}^{N} \lambda_\nu \frac{g(z)}{z-z_\nu}$$

where λ_ν are non-negative reals with

$$\sum_{\substack{\nu=1 \\ z_\nu \neq 0}}^{N} \lambda_\nu = 1,$$

α is a complex number, and $P(z)$ is a polynomial of degree $\leqq s-2$ such that $P(z)\,g(z) + \alpha z^{K+N-2}$ is of degree $\leqq n$.

IV. Similarly, with the aid of Theorem 2 we can show the following. Let hypotheses 1—3 of Theorem 1 and hypothesis 2 of Theorem 2 hold, let $0 \in \sigma$, $0 \in S$, and let $a_0^* = 0$.

a. If $N = n-s+2$, $\alpha_0(z)$ (see below) is the unique n-th infrapolynomial on S with respect to σ, belonging to Π.

b. If $N = n-s+3$, a polynomial $A(z)$ is an n-th infrapolynomial on S with respect to σ, belonging to Π, if, and only if, it is of the form

$$\alpha_0(z) + \Lambda_{N-2}\, z^{K-1} \sum_{\substack{\nu=1 \\ z_\nu \neq 0}}^{N} \lambda_\nu \frac{g(z)}{z-z_\nu}$$

where λ_ν are non-negative integers with

$$\sum_{\substack{\nu=1 \\ z_\nu \neq 0}}^{N} \lambda_\nu = 1 .$$

Here $\alpha_0(z) \equiv (\sum_{\nu \in \sigma} a_\nu^* z^\nu) - z^K \Lambda(z)$, where $\Lambda(z) \equiv \Lambda_{N-2} z^{N-2} + \dots + \Lambda_0$ is Lagrange's interpolation polynomial to $z^{-K} \sum_{\nu \in \sigma} a_\nu^* z^\nu$ on $S - \{0\}$.

§ 2. On the zeros of n-th infrapolynomials with respect to $(0, n)$.

Many results on such polynomials $\sum_{\nu=0}^{n} a_\nu z^\nu$ (with $a_n = 1$) were given recently in a paper devoted primarily to their study [Walsh 1958].

We begin by considering the location of the zeros of n-th infrapolynomials on sets of n points with respect to $(0, n)$.

Theorem 7. *Let n be an integer $\geqq 2$, $S (\not\ni 0)$ a set of n distinct points of the z-plane: $S = \{z_1, z_2, \dots, z_n\}$, and $A(z) \equiv a_0^* + a_1 z + \dots + a_{n-1} z^{n-1} + a_n^* z^n$ an n-th infrapolynomial on S with respect to $(0, n)$. Set*

$$g(z) \equiv \prod_{\nu=1}^{n} (z - z_\nu) .$$

I. *If $a_n^* = 0$, $a_0^* \neq 0$,*(11) *then every disc*(12) *Γ such that $S \subset \Gamma$, $0 \notin \Gamma$, contains all zeros of $A(z)$.*(13)

II. *Suppose $a_n^* \neq 0$, and denote $\alpha^* = \dfrac{a_0^*}{a_n^* \, g(0)}$.*

a) *If $\alpha^* = 1$, $A(z) \equiv \dfrac{a_0^*}{g(0)} g(z)$.*

b) *Let α^* be $\neq 1$, and let Γ be a disc containing S. Then all zeros of $A(z)$ belong to $\Gamma \cup \Gamma'$, where Γ' is the disc $\alpha^* \Gamma$ [i.e., the set of all $\alpha^* z$, $z \in \Gamma$].*(14) *If Γ and Γ' are disjoint, the number of zeros of $A(z)$ in Γ and in Γ' is, respectively, $n-1$ and 1.*(15)

11. If $a_0^* = a_n^* = 0$, then by (18) (below), $A(z) \equiv 0$.

12. A "disc" will mean a set of the form $\{z : |z - \gamma_0| \leqq r\}$, where $0 \leqq r < \infty$.

13. If $a_n^* = 0$, $A(z)$ is easily seen to be an $(n-1)$-th infrapolynomial on S with respect to (0). By Theorem 10 (below), if $a_0^* \neq 0$, all zeros of $A(z)$ belong to the "convex hull of S with respect to 0".

14. Obviously, if Γ has center γ_0 and radius r, $\alpha^* \Gamma$ will have center $\alpha^* \gamma_0$ and radius $|\alpha^*| r$.

15. When we speak of "number of zeros", multiplicities are always counted.

We shall need in the proof the following theorem [Walsh, 1958, Corollary 1 to Theorem 8].

Theorem. *Let $z_1, z_2, \ldots, z_n$ be points of the z-plane lying in a disc Γ. Let C be a complex number, $\lambda_1, \lambda_2, \ldots, \lambda_n$ non-negative reals with $\sum_{\nu=1}^{n}\lambda_\nu = 1$, and let*

$$R(z) \equiv C - \sum_{\nu=1}^{n} \lambda_\nu \frac{z}{z-z_\nu}.$$

Then

I. *If $C=1$ and $0 \notin \Gamma$, all zeros of $R(z)$ belong to Γ.*

II. *Suppose $C \neq 1$. Then all zeros of $R(z)$ belong to $\Gamma \cup \Gamma'$ where Γ' is the disc $\frac{C}{C-1}\Gamma$. If Γ and Γ' are disjoint, then the number of zeros of $R(z)$ in Γ' is 1.*

Proof of Theorem 7. We set, in accordance with (12)

$$A(z) \equiv \frac{a_0^*}{g(0)} g(z) + \left(a_n^* - \frac{a_0^*}{g(0)}\right) z \sum_{\nu=1}^{n} \lambda_\nu \frac{g(z)}{z-z_\nu}, \tag{18}$$

where the λ_ν's are certain non-negative reals with $\sum_{\nu=1}^{n} \lambda_\nu = 1$.

I. Let $a_n^* = 0$, $a_0^* \neq 0$, and let $\Gamma\,(\not\ni 0)$ be a disc containing S. Let z_0 be a zero of $A(z)$. If $g(z_0) = 0$, $z_0 \in S$, and therefore $z_0 \in \Gamma$. If $g(z_0) \neq 0$, then by (18)

$$1 - \sum_{\nu=1}^{n} \lambda_\nu \frac{z_0}{z_0 - z_\nu} = 0,$$

and by I of the previous theorem, $z_0 \in \Gamma$ again.

II. Let $a_n^* \neq 0$. If $\alpha^* = 1$, then by the definition of α^* and by (18), $A(z) \equiv \frac{a_0^*}{g(0)} g(z)$. Suppose, now, that $\alpha^* \neq 1$ and that Γ is a disc containing S. From (18) we get

$$A(z) \equiv \left[\frac{a_0^*}{g(0)} - a_n^*\right] \left[C g(z) - z \sum_{\nu=1}^{n} \lambda_\nu \frac{g(z)}{z-z_\nu}\right] \tag{19}$$

where $C = \dfrac{a_0^*}{a_0^* - a_n^* g(0)} = \dfrac{\alpha^*}{\alpha^* - 1}$, and therefore $\dfrac{C}{C-1} = \alpha^*$.

Let

$$R(z) \equiv C - \sum_{\nu=1}^{n} \lambda_\nu \frac{z}{z - z_\nu}. \tag{20}$$

By the previous theorem, all zeros of $R(z)$ belong to $\Gamma \cup \Gamma'$. Since the same is obviously true of $g(z)$, all zeros of $A(z)$ belong to $\Gamma \cup \Gamma'$. If Γ and Γ' are disjoint, then the number of zeros of $A(z)$ in Γ' is the same as that of $R(z)$, which by the previous theorem, is 1. This concludes the proof.

Theorem 7 will be generalized (to arbitrary sets S) in Theorems 11 and 12 below.

The next theorem relates the zeros of an n-th infrapolynomial on a set S of n points with respect to $(0, n)$, to the convex hull of S (rather than to a disc containing S, as in Theorem 7). For this purpose we need the following

Lemma. Let $z_1, z_2, \ldots, z_n$ be points of the z-plane, $\lambda_1, \lambda_2, \ldots, \lambda_n$ non-negative reals with $\sum_{\nu=1}^{n} \lambda_\nu = 1$, and C an arbitrary complex number. Let

$$S = \{z_1, z_2, \ldots, z_n\},$$

$$g(z) \equiv \prod_{\nu=1}^{n} (z - z_\nu),$$

and

$$P(z) \equiv C g(z) - z \sum_{\nu=1}^{n} \lambda_\nu \frac{g(z)}{z - z_\nu}.$$

Let ζ be any zero of $P(z)$. Then:

I. If $C = 0$, $\zeta \in \{0\} \cup S^*$.

II. If $C \leq 0$, $\zeta \in (\{0\} \cup S)^*$.

III. In each of the following cases:

1. $C > 0$, $0 \notin S^*$,
2. $C < 0$ or $C > 1$,

3. C is not real,

ζ can be written in the form ρc^* with $c^* \in S^*$ and with ρ satisfying:

In case 1: $\rho < 0$ or $\rho \geq 1$.[16]

In case 2: ρ (is positive and) belongs to the closed interval whose endpoints are 1 and $\dfrac{C}{C-1}$.

In case 3: ρ belongs to the image T of the closed interval $[0, 1]$ under the transformation $w = \dfrac{C}{C-z}$. T is a circular arc and can be represented as

$$T = \left\{ -\frac{ie^{i\gamma}}{2\sin\gamma} + \frac{1}{2\sin\gamma} e^{it} : \ t \in I_1 \right\}.$$

Also

$$T = \left\{ \frac{\sin(\gamma - t)}{2\sin\gamma} e^{it} : \ t \in I_2 \right\}. \tag{21}$$

I_1 is the closed interval whose endpoints are $\dfrac{\pi}{2} - \gamma$ and $\dfrac{\pi}{2} - \gamma + 2\gamma_0$; I_2 is the closed interval whose endpoints are 0 and γ_0. γ and γ_0 are, respectively, the arguments of C and of $\dfrac{C}{C-1}$ (with $|\gamma| < \pi$, $|\gamma_0| < \pi$). As a result of (21),

$$|\rho| \leq \max \left[\left| \frac{\sin(\gamma - t)}{\sin\gamma} \right|, \ t \text{ on } I_2 \right] \leq \frac{1}{|\sin\gamma|} = \frac{|C|}{|\operatorname{Im}(C)|}.$$

Proof. I, II and III are trivially true if $\zeta \in S$ (in III we can take then $c^* = \zeta$, $\rho = 1$). We assume, therefore, that $\zeta \notin S$, and so

$$C - \zeta \sum_{\nu=1}^{n} \frac{\lambda_\nu}{\zeta - z_\nu} = 0. \tag{22}$$

We make use of the well-known fact that an equality of the form

16. Therefore, every set in the z-plane containing S^*, and containing together with each of its points all its real multiples, contains ζ.

$$\sum_{\nu=1}^{q} \frac{\mu_\nu}{x-x_\nu} = 0 ,$$

where $x , x_1 , \dots , x_q$ are complex and μ_ν are non-negative reals with

$$\sum_{\nu=1}^{q} \mu_\nu > 0 ,$$

implies that x lies in $\{x_1 , x_2 , \dots , x_q\}^*$.

From this and from (22), I follows immediately, and so does II, for, if $C \leqq 0$ and $\zeta \neq 0$,

$$\frac{|C|}{\zeta-0} + \sum_{\nu=1}^{n} \frac{\lambda_\nu}{\zeta-z_\nu} = 0 .$$

To prove III we set, in accordance with the lemma preceding the proof of Theorem 6,

$$\sum_{\nu=1}^{n} \frac{\lambda_\nu}{\zeta-z_\nu} = \frac{\lambda}{\zeta-c^*} \qquad (0 \leqq \lambda \leqq 1,\ c^* \in S^*) .$$

From (22) we get

$$C - \frac{\lambda\zeta}{\zeta-c^*} = 0 , \quad \zeta (C-\lambda) = Cc^* .$$

In each of the cases 1, 2, 3 of III, $\lambda \neq 0$, $C-\lambda \neq 0$, and by setting

$$\rho = \frac{C}{C-\lambda} \tag{23}$$

we have $\zeta = \rho c^*$. Moreover, in case 1, $\rho < 0$ or $\rho > 1$ (depending on whether $C < \lambda$ or $C > \lambda$); in case 2, $1 \leqq \rho \leqq \dfrac{C}{C-1}$ if $C > 1$, and $\dfrac{C}{C-1} \leqq \rho \leqq 1$ if $C < 0$. Finally, in case 3, (23) shows that ρ belongs to the image of the closed interval $[0 , 1]$ under $\dfrac{C}{C-z}$. The rest of the lemma is obtained by some elementary considerations which we may omit. Compare Walsh [1950, § 4.2.3].

Theorem 8. *Let $S\,(\not\ni 0)$ be a set of $n (\geqq 2)$ distinct points of the z-plane: $S = \{z_1 , z_2 , \dots , z_n\}$. Let*

$$A(z) \equiv a_0^* + a_1 z + \dots + a_{n-1} z^{n-1} + a_n^* z^n$$

be an n-th infrapolynomial on S with respect to $(0, n)$. Suppose $a_0^* - a_n^* g(0) \neq 0$ [17] where

$$g(z) \equiv \prod_{\nu=1}^{n} (z - z_\nu),$$

and denote $C = \dfrac{a_0^*}{a_0^* - a_n^* g(0)}$. Then for every zero ζ of $A(z)$, I, II and III of the preceding lemma hold.

Proof. We set, in accordance with (12)

$$A(z) \equiv \frac{a_0^*}{g(0)} g(z) + \left(a_n^* - \frac{a_0^*}{g(0)}\right) z \sum_{\nu=1}^{n} \lambda_\nu \frac{g(z)}{z - z_\nu},$$

where the λ_ν are certain non-negative reals with

$$\sum_{\nu=1}^{n} \lambda_\nu = 1 .$$

Then

$$A(z) \equiv \frac{a_0^* - a_n^* g(0)}{g(0)} P(z),$$

where

$$P(z) \equiv C g(z) - z \sum_{\nu=1}^{n} \lambda_\nu \frac{g(z)}{z - z_\nu}.$$

The conclusion of our theorem follows now from the last lemma.

As a direct consequence of our last lemma we have also the following result on the location of the zeros of polynomials of the form $A g(z) - z g'(z)$.

Theorem 9. Let

$$g(z) \equiv \prod_{\nu=1}^{n} (z - z_\nu) \qquad (n \geq 1)$$

be a polynomial, C an arbitrary complex number,

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17. If $a_0^* - a_n^* g(0) = 0$, then by (12), $A(z) \equiv \dfrac{a_0^*}{g(0)} g(z)$.

$$S = \{z_1, z_2, \ldots, z_n\},$$

and

$$h(z) \equiv nCg(z) - zg'(z).$$

Let ζ be any zero of $h(z)$. Then I, II and III of the last lemma hold.

Proof. The theorem follows at once from the last lemma upon noticing that

$$h(z) \equiv n\left\{Cg(z) - z\sum_{\nu=1}^{n}\frac{1}{n}\frac{g(z)}{z-z_\nu}\right\}.$$

Before turning to the study of infrapolynomials on arbitrary (not necessarily finite) sets, with respect to $(0, n)$, we consider a narrower class of n-th infrapolynomials, namely, that of n-th infrapolynomials with respect to (0). Such infrapolynomials are related in a simple way to n-th infrapolynomials with respect to (n). This relation, essentially due to M. Fekete, is as follows.

Lemma. Let S be a set in the z-plane, $0 \notin S$, and let

$$A(z) \equiv \sum_{\nu=0}^{n} a_\nu z^\nu \qquad (n \geqq 1)$$

be a polynomial. Then $A(z)$ is an n-th infrapolynomial on S with respect to (0), if, and only if,

$$\tilde{A}(z) \equiv \sum_{\nu=0}^{n} a_{n-\nu} z^\nu$$

is an n-th infrapolynomial on S^{-1} with respect to (n). (S^{-1} is the set of all z^{-1}, $z \in S$).

Proof. Let $A(z)$ be an n-th infrapolynomial on S with respect to (0). Suppose $\tilde{A}(z)$ is not an n-th infrapolynomial on S^{-1} with respect to (n), and let

$$\tilde{B}(z) \equiv \sum_{\nu=0}^{n} b_{n-\nu} z^\nu$$

be an n-th underpolynomial of $\tilde{A}(z)$ on S^{-1} with respect to (n). Using the

fact that

$$\tilde{A}(z) \equiv z^n A(z^{-1}), \ \tilde{B}(z) \equiv z^n B(z^{-1}), \ \left[B(z) \equiv \sum_{\nu=0}^{n} b_\nu z^\nu\right],$$

one easily shows that $B(z)$ is an n-th underpolynomial of $A(z)$ on S with respect to (0), which contradicts our hypothesis.

This proves one half of the lemma. The other half is similarly proved.

More generally: let $n, n_1, n_2, \ldots, n_q$ be integers, $1 \leqq q \leqq n$, $0 \leqq n_1 < n_2 < \ldots < n_q \leqq n$, let S be a set in the z-plane, and $A(z)$ an n-th infrapolynomial on S with respect to $\tau = (n_1, n_2, \ldots, n_q)$. If $0 \notin S$, then $z^n A(z^{-1})$ is easily seen to be an n-th infrapolynomial on S^{-1} with respect to $(n-n_q, n-n_{q-1}, \ldots, n-n_1)$. Also: let α be an arbitrary complex number. One shows readily that if $\alpha \neq 0$, then $A(\alpha z)$ is an n-th infra-polynomial on $\alpha^{-1} S$ with respect to τ, and that if τ is of the form $(n-l, n-l+1, \ldots, n)$ $[l \geqq 0]$, then $A(z-\alpha)$ is an n-th infrapolynomial on $\alpha + S$ with respect to τ. (We denote by $\alpha + S$ the set of all $\alpha + z$, $z \in S$).

Before using the last lemma, we make the following observation. Let S be a set in the z-plane such that 0 does not belong to $\overline{S}$, the closure of S. Let C be the intersection of all circular regions R such that $R \supseteq S$, $0 \notin R$. (Circular regions are discs, closures of complements of discs with respect to the z-plane, and closed half-planes). Let D be the intersection of all discs containing S^{-1}. It is not difficult to see that

$$C = [D - \{0\}]^{-1} = [(\overline{S^{-1}})^* - \{0\}]^{-1}$$

(Cf. footnote 4 of the introduction). C is called *the convex hull of S with respect to 0.*

Using this concept we have the following hitherto unpublished theorem due to M. Fekete:

Theorem 10. *Let S be a set in the z-plane, $0 \notin \overline{S}$. Let n be an arbitrary natural number,*

$$A(z) \equiv \sum_{\nu=0}^{n} a_\nu z^\nu$$

an n-th infrapolynomial on S with respect to (0), and let

$a_0 = A(0) \neq 0$.[18] Then all zeros of $A(z)$ belong to C, the convex hull of S with respect to 0.

Corollary. If S lies in a circular region R which does not contain 0, then all zeros of $A(z)$ belong to R. (For, obviously, $C \subseteq R$).

Proof of Theorem 10. By the last lemma,

$$\tilde{A}(z) \equiv \sum_{\nu=0}^{n} a_{n-\nu} z^{\nu}$$

is an n-th infrapolynomial on S^{-1} with respect to (n). $\tilde{A}(z)$ is a fortiori an n-th infrapolynomial on $\overline{S^{-1}}$ (containing S^{-1}) with respect to (n). Therefore, by a well known theorem of Fejér [Fejér, 1922], all zeros of $\tilde{A}(z)$ belong to $(\overline{S^{-1}})^*$. If z_0 is a zero of $A(z)$, then $z_0 \neq 0$,

$$\tilde{A}(z_0^{-1}) = z_0^n A(z_0) = 0,$$

and therefore, $z_0^{-1} \in (\overline{S^{-1}})^* - \{0\}$, which implies that

$$z_0 \in [(\overline{S^{-1}})^* - \{0\}]^{-1} = C.$$

We turn now to n-th infrapolynomials on general sets S with respect to $(0, n)$. The following Theorem 11 was motivated by an earlier, but weaker result by J. L. Walsh [1958, Theorem 9].

Theorem 11. Let S be a subset of a disc Δ $(0 \notin \Delta)$, and let $A(z)$ $(\not\equiv 0)$ be an n-th $(n \geq 2)$ infrapolynomial on S with respect to $(0, n)$. Then there exists a complex number α^* such that

a. All zeros of $A(z)$ belong to $\Delta \cup \alpha^* \Delta$.

b. If Δ and $\alpha^* \Delta$ are disjoint, the number of zeros of $A(z)$ in $\alpha^* \Delta$ is at most 1.

Remark. For every disc Δ not containing 0, and every complex number α $(\neq 0)$, Δ and $\alpha\Delta$ subtend the same angle at 0 [cf. Walsh 1958, §5].

In the proof of Theorem 11 (and of some other theorems of §§ 2, 3) we use the following structure theorem due to O. Shisha [1961], which is a generalization of a previous theorem of M. Fekete [1951, Theorem 1].

18. $a_0 \neq 0$ is easily seen to be equivalent to $A(z) \not\equiv 0$.

Theorem. Let n and s $(1 \leqq s \leqq n)$ be integers, and σ a simple n-sequence of s elements. Let S be a compact set in the z-plane, and in case $0 \in \sigma$, assume $0 \notin S$. Let $A(z)$ $(\not\equiv 0)$ be an n-th infrapolynomial on S with respect to σ, and let $B(z)$ $(\neq 0$ throughout $S)$ be a divisor of $A(z)$ of degree $r \geqq s$. Then $B(z)$ is a divisor of some

$$Q(z) \equiv P(z)\, g(z) + z^K \sum_{\nu=1}^{m-s+2} \lambda_\nu \frac{g(z)}{z - z_\nu},$$

an m-th infrapolynomial on $\{z_1, z_2, \ldots, z_{m-s+2}\} \subseteq S$ with respect to σ_1. Here m is an integer satisfying $r \leqq m \leqq 2r - s + 1$, the z_ν are distinct points of S,

$$g(z) \equiv \prod_{\nu=1}^{m-s+2} (z - z_\nu),$$

the λ_ν are positive reals with

$$\sum_{\nu=1}^{m-s+2} \lambda_\nu = 1,$$

$P(z)$ is a polynomial of degree $\leqq s - 1$ such that $P(z)\, g(z) + z^{K+m-s+1}$ is of degree $\leqq m$, K is $\min [\nu, \nu \notin \sigma, \nu = 0, 1, 2, \ldots]$ and σ_1 is that simple m-sequence of s elements for which $\min [\nu, \nu \notin \sigma_1, \nu = 0, 1, 2, \ldots] = K$.

Proof of Theorem 11. Let $A(z) \equiv B(z)\, D(z)$, where $D(z)$ is a polynomial all of whose zeros belong to $\overline{S}$, and where

$$B(z) \left(\equiv \sum_{\nu=0}^{r} b_\nu z^\nu,\ b_r \neq 0\right)$$

is a polynomial which is $\neq 0$ throughout $\overline{S}$.

Our theorem is trivial if $r = 0$. If $r = 1$, let $B(z) \equiv b_r (z - \zeta)$, and choose an arbitrary point d of Δ. Then $\alpha^* = \dfrac{\zeta}{d}$ fulfills (a) and (b) of the Theorem.

We assume, therefore, $r \geqq 2$. $A(z)$ is obviously an n-th infrapolynomial on $\overline{S}$ with respect to $(0, n)$ (since $\overline{S} \supseteq S$). By the structure theorem stated

above (with our $\bar{S}$ in place of S of that theorem), $B(z)$ is a divisor of some

$$Q(z) \equiv \alpha g(z) + z \sum_{\nu=1}^{m} \lambda_\nu \frac{g(z)}{z-z_\nu},$$

where α is a complex number,

$$g(z) \equiv \prod_{\nu=1}^{m} (z - z_\nu),$$

all $z_\nu \in \bar{S}$, and where the λ_ν are positive numbers with

$$\sum_{\nu=1}^{m} \lambda_\nu = 1.$$

Setting $\alpha^* = 1$ if $\alpha = -1$, and $\alpha^* = \dfrac{\alpha}{\alpha+1}$ if $\alpha \neq -1$, we get by Walsh's Theorem stated after Theorem 7, that all zeros of $B(z)$ belong to $\Delta \cup (\alpha^* \Delta)$, and that if Δ and $\alpha^* \Delta$ are disjoint, the number of zeros of $B(z)$ in $\alpha^* \Delta$ is at most 1. From this Theorem 11 follows.

The following is a modification of Theorem 11.

Theorem 12.

Hypotheses:

1. S is a subset of a disc Δ, $0 \notin \bar{S}$.
2. $A(z) \equiv \sum_{\nu=0}^{n} a_\nu z^\nu$ $(n \geqq 2,\ a_n \neq 0)$ is an n-th infrapolynomial on S with respect to $(0, n)$, but is not an n-th infrapolynomial on S with respect to (0).[19]
3. Every zero of $A(z)$ which belongs to $\bar{S}$ is a limit point of S (this is fulfilled, e.g., if $A(z)$ is $\neq 0$ throughout S, or if $\bar{S}$ is a perfect set).

Conclusion: Same as that of Theorem 11.

For the proof of Theorem 12 we need the following

Lemma. Let S $(0 \notin \bar{S})$ be a set in the z-plane, and let

19. If S is a subset of a disc Δ, if $0 \notin \Delta$, and if $A(z)$ $(\not\equiv 0)$ is an n-th $(n \geqq 1)$ infrapolynomial on S with respect to (0), then by the corollary to Theorem 10, all zeros of $A(z)$ belong to Δ.

$$C(z) \equiv \sum_{\nu=0}^{n} c_\nu z^\nu \qquad (n \geq 1)$$

be a polynomial.

a. $C(z)$ is an n-th infrapolynomial on S with respect to (0) if and only if it is an n-th infrapolynomial on $\overline{S}$ with respect to (0).

b. If $C(z)$ is an n-th infrapolynomial on S with respect to (0), and if

$$D(z) \equiv \sum_{\nu=0}^{p} d_\nu z^\nu \qquad (d_p = 1)$$

is a polynomial all of whose zeros belong to $S \cap S'$ (S' being the derived set of S), then $A(z) \equiv C(z)\, D(z)$ is an $(n+p)$-th infrapolynomial on S with respect to (0).

Proof of the Lemma.

a. Obviously, if $C(z)$ is an n-th infrapolynomial on S with respect to (0), it is also an n-th infrapolynomial on $\overline{S}$ with respect to (0). Suppose $C(z)$ is an n-th infrapolynomial on $\overline{S}$ with respect to (0). We shall show that $C(z)$ is an n-th infrapolynomial on S with respect to (0). We may assume that S is infinite (otherwise $S = \overline{S}$). If $c_0 = 0$, then $C(z) \equiv 0$ (cf. footnote (18) of this section) and, therefore, $C(z)$ is an n-th infrapolynomial on S with respect to (0). We thus suppose that $c_0 \neq 0$. By the Lemma following Theorem 9, $\tilde{C}(z) \equiv c_0 z^n + \ldots + c_n$ is an n-th infrapolynomial on $(\overline{S})^{-1}$ $(= \overline{S^{-1}})$ with respect to (n). This implies [cf. Motzkin and Walsh 1957a, Corollary, p. 410] that $\tilde{C}(z)$ is an n-th infrapolynomial on S^{-1} with respect to (n). Using again the same Lemma, we get that $C(z)$ is an n-th infrapolynomial on S with respect to (0).

b. Let $C(z)$ and $D(z)$ satisfy the hypotheses of part (b) of the present Lemma. Let $\tilde{C}(z)$ be as before,

$$\tilde{D}(z) \equiv \sum_{\nu=0}^{p} d_{p-\nu} z^\nu, \quad A(z) \equiv \sum_{\nu=0}^{n+p} a_\nu z^\nu \quad \text{and} \quad \tilde{A}(z) \equiv \sum_{\nu=0}^{n+p} a_{n+p-\nu} z^\nu .$$

From $A(z) \equiv C(z)\, D(z)$ it follows that $\tilde{A}(z) \equiv \tilde{C}(z)\, \tilde{D}(z)$. We may assume

that $p>0$, which implies that S' is not empty, and that therefore S is infinite. As before, we can assume that $c_0 \neq 0$, since otherwise $A(z) \equiv C(z) \equiv 0$. It follows from our hypotheses regarding $D(z)$, that all zeros of $\tilde{D}(z)$ belong to $(S^{-1}) \cap (S^{-1})'$. Also $d_0 \neq 0$, for otherwise $D(0)=0$ although $0 \notin S$. Since $\tilde{C}(z)$ is an n-th infrapolynomial on S^{-1} (and therefore on $\overline{S^{-1}}$) with respect to (n), and since every zero of $\tilde{D}(z)$ belongs to $(\overline{S^{-1}}) \cap (\overline{S^{-1}})'$, then by the second part of Theorem 2 of the paper last mentioned, and by the Corollary on p. 410 of that paper, $\tilde{A}(z)$ is an $n+p$-th infrapolynomial on S^{-1} with respect to (n). Hence the desired conclusion follows.

Proof of Theorem 12. We set again

$$A(z) \equiv B(z)\, D(z),$$

$$B(z) \equiv \sum_{\nu=0}^{r} b_\nu z^\nu \quad (b_r = a_n), \qquad D(z) \equiv \sum_{\nu=0}^{n-r} d_\nu z^\nu \quad (d_{n-r}=1),$$

where all the zeros of $D(z)$ belong to $\overline{S}$, while no zero of $B(z)$ belongs to $\overline{S}$. By hypothesis 3, every zero of $D(z)$ belongs to $\overline{S} \cap (\overline{S})'$.

We may assume, as in the proof of Theorem 11, that $r \geqq 2$. $B(z)$ is not an r-th infrapolynomial on $\overline{S}$ with respect to (0). For otherwise, by parts (b) and (a) of the last Lemma (with $B(z)$ and $A(z)$ taken in place of $C(z)$) $A(z)$ would be an n-th infrapolynomial on S with respect to (0), contradicting hypothesis 2.

We use now the structure theorem following the statement of Theorem 11 (with our $\overline{S}$ replacing S). By that theorem $B(z)$ is a divisor of some

$$Q(z) \equiv \alpha g(z) + z \sum_{\nu=1}^{m} \lambda_\nu \frac{g(z)}{z-z_\nu} \equiv (\alpha+1) z^m + \dots \qquad (m \geqq 2),$$

an m-th infrapolynomial on $\{z_1, z_2, \dots, z_m\} \subseteq \overline{S}$ with respect to $(0, m)$. If $\alpha+1$ were zero, then $Q(z)$ would be an $(m-1)$-th infrapolynomial on $\overline{S}$ with respect to (0). This, however, would imply, as is easily seen, that $B(z)$ is an r-th infrapolynomial on $\overline{S}$ with respect to (0), contradicting the conclusion just made. Thus $-\alpha \neq 1$.

By Walsh's theorem stated after Theorem 7, every zero of $B(z)$

belongs to $\Delta \cup (\alpha^* \Delta)$, where $\alpha^* = \dfrac{\alpha}{\alpha+1}$, and if Δ and $\alpha^* \Delta$ are disjoint, the number of zeros of $B(z)$ in the latter disc is ≤ 1. From this our theorem follows.

Theorem 13. *Let the hypotheses of Theorem 11 hold. Let c be the center of Δ, and r its radius. Then: either all zeros of $A(z)$ $\left(\equiv \sum_{\nu=0}^{n} a_\nu z^\nu\right)$ belong to Δ except, perhaps, for one, ζ, which is simple and satisfies:*

$$|\zeta| \leq |a_0| \, [|a_n| \, (|c| - r)^{n-1}]^{-1}, \tag{23a}$$

or there exists an α^*,

$$(|c| - r)\,(|c| + r)^{-1} \leq |\alpha^*| \leq (|c| + r)\,(|c| - r)^{-1}, \tag{23b}$$

such that all zeros of $A(z)$ belong to $\Delta \cup \alpha^* \Delta$.

Proof. Let α^* be a number satisfying (a) and (b) of Theorem 11.

I. Suppose Δ and $\alpha^* \Delta$ are disjoint. If not all the zeros of $A(z)$ belong to Δ, let ζ be the unique zero of $A(z)$ which does not belong to Δ, so that

$$A(z) \equiv a_n (z - \zeta) \prod_{\nu=1}^{n-1} (z - z_\nu) \qquad (z_\nu \in \Delta, \ \nu = 1, 2, \ldots, n-1).$$

We have

$$|a_0| = |a_n| \, |\zeta| \prod_{\nu=1}^{n-1} |z_\nu| = |a_n| \, |\zeta| \prod_{\nu=1}^{n-1} |c - (z_\nu - c)| \geq |a_n| \, |\zeta| \, (|c| - r)^{n-1},$$

and (23a) holds.

II. Suppose Δ and $\alpha^* \Delta$ are not disjoint. Then $|\alpha^* c - c| \leq |\alpha^*| r + r$ and therefore:

$$|c| \, (|\alpha^*| - 1) \leq |c| \, |\alpha^* - 1| \leq r \, (|\alpha^*| + 1),$$
$$|c| \, (1 - |\alpha^*|) \leq |c| \, |\alpha^* - 1| \leq r \, (|\alpha^*| + 1);$$

hence

$$|\alpha^*| \, (|c| - r) \leq |c| + r, \quad |\alpha^*| \, (|c| + r) \geq |c| - r,$$

from which (23b) follows.

§3. On the zeros of n-th infrapolynomials with respect to $(n-1, n)$

Many properties of such polynomials $z^n + ... + a_0$ (and of related and more general ones) were given in various earlier papers [for instance, Zedek 1955, Walsh and Zedek 1956, and Fekete and Walsh 1957].

We begin by considering n-th infrapolynomials on sets of n points with respect to $(n-1, n)$.

Lemma. Let $S = \{z_1, z_2, ..., z_n\}$ ($n \geqq 2$, $z_\alpha \neq z_\beta$ for $\alpha \neq \beta$) be a finite set in the z-plane, and let a_n^* and a_{n-1}^* be given complex numbers. A necessary and sufficient condition for a polynomial $A(z)$ to be an n-th infrapolynomial on S with respect to $(n-1, n)$ of the form

$$a_0 + ... + a_{n-2} z^{n-2} + a_{n-1}^* z^{n-1} + a_n^* z^n,$$

is the existence of non-negative reals $\lambda_1, \lambda_2, ..., \lambda_n$ with

$$\sum_{\nu=1}^{n} \lambda_\nu = 1$$

such that

$$A(z) \equiv a_n^* g(z) + \left(a_{n-1}^* + a_n^* \sum_{\nu=1}^{n} z_\nu\right) \sum_{\nu=1}^{n} \lambda_\nu \frac{g(z)}{z - z_\nu},$$

where

$$g(z) \equiv \prod_{\nu=1}^{n} (z - z_\nu).$$

The Lemma follows easily from Theorem 1.

Theorem 14. Let n be a natural number $\geqq 2$,

$$S = \{z_1, z_2, ..., z_n\}$$

a set of n (distinct) points of the z-plane, contained in a disc Γ. Let $A(z) \equiv a_0 + a_1 z + ... + a_{n-2} z^{n-2} + a_{n-1}^* z^{n-1} + a_n^* z^n$ be an n-th infrapolynomial on S with respect to $(n-1, n)$, and suppose $a_n^* \neq 0$.[20] Set

20. If $a_n^* = 0$, then by the last Lemma,

$$A(z) \equiv a_{n-1}^* \sum_{\nu=1}^{n} \lambda_\nu \frac{g(z)}{z - z_\nu} \quad \left(\lambda_\nu \geqq 0, \ \sum_{\nu=1}^{n} \lambda_\nu = 1\right).$$

In this case, if $a_{n-1}^* = 0$, $A(z) \equiv 0$, while if $a_{n-1}^* \neq 0$, all zeros of $A(z)$ belong to S^* and, therefore, to Γ.

$$\gamma^* = -\frac{a^*_{n-1}}{a^*_n} - \sum_{\nu=1}^{n} z_\nu .$$

Then all zeros of $A(z)$ belong to $\Gamma \cup (\gamma^*+\Gamma)$,[21] and if Γ and $\gamma^*+\Gamma$ are disjoint, the number of zeros of $A(z)$ in them is, respectvely, $n-1$ and 1.

Theorem 14 follows from the following Theorem due to J. L. Walsh [cf. 1922, Theorem VI] and from the last Lemma.

Theorem. Let $z_1, \dots, z_n$ be points of the z-plane lying in a disc Γ. Let γ be a complex number and $\lambda_1, \lambda_2, \dots, \lambda_n$ non-negative reals with

$$\sum_{\nu=1}^{n} \lambda_\nu = 1 .$$

Then all zeros of

$$g_1(z) \equiv g(z) - \gamma \sum_{\nu=1}^{n} \lambda_\nu \frac{g(z)}{z-z_\nu}$$

belong to $\Gamma \cup (\gamma+\Gamma)$. If Γ and $\gamma+\Gamma$ are disjoint, the number of zeros of $g_1(z)$ in them is, respectively, $n-1$ and 1

$$\left(g(z) \equiv \prod_{\nu=1}^{n} (z-z_\nu)\right).$$

For the next theorem we need the following

Lemma. Let $z_1, z_2, \dots, z_n$ be points of the z-plane, γ a complex number, $\lambda_1, \lambda_2, \dots, \lambda_n$ non-negative reals with

$$\sum_{\nu=1}^{n} \lambda_\nu = 1 ,$$

and let z_0 be a zero of

$$g_1(z) \equiv g(z) - \gamma \sum_{\nu=1}^{n} \lambda_\nu \frac{g(z)}{z-z_\nu} ,$$

---

21. If Γ has center c and radius r, $\gamma^*+\Gamma$ is the disc with center γ^*+c and radius r.

where

$$g(z) \equiv \prod_{\nu=1}^{n} (z - z_\nu) .$$

Then z_0 has the form $c + \lambda\gamma$ where $c \in \{z_1, z_2, \ldots, z_n\}^*$, and where $0 \leqq \lambda \leqq 1$.

Remark. In case each λ_ν is $\frac{1}{n}$ (so that $g_1(z) \equiv g(z) - \frac{\gamma}{n} g'(z)$), this result was given by T. Takagi [1921. See also Walsh 1924].

Proof of the Lemma. We may assume that z_0 is different from all z_ν ($\nu = 1, 2, \ldots, n$). By the Lemma following the statement of Theorem 6, we have

$$0 = 1 - \gamma \frac{\lambda}{z_0 - c} \qquad (c \in \{z_1, z_2, \ldots, z_n\}^*, \ 0 \leqq \lambda \leqq 1),$$

and the result follows.

Theorem 15. Let n be a natural number $\geqq 2$,

$$S = \{z_1, z_2, \ldots, z_n\}$$

a set of n (distinct) points of the z-plane, and

$$A(z) = a_0 + \ldots + a_{n-2} z^{n-2} + a_{n-1}^* z^{n-1} + a_n^* z^n \qquad (a_n^* \neq 0)$$

an n-th infrapolynomial on S with respect to $(n-1, n)$. Then every zero z of $A(z)$ is of the form $c + \lambda\gamma^*$ where

$$c = c(z) \in S^*, \ \lambda = \lambda(z)$$

satisfies $0 \leqq \lambda \leqq 1$ and where (as in Theorem 14)

$$\gamma^* = -\frac{a_{n-1}}{a_n^*} - \sum_{\nu=1}^{n} z_\nu . \text{ [22]}$$

This theorem follows from the last two lemmas.

We proceed now to the study of n-th infrapolynomials on general sets with respect to $(n-1, n)$.

Theorem 16. Let S be a subset of a disc Δ, and let $A(z)\,(\not\equiv 0)$ be an n-th ($n \geqq 2$) infrapolynomial on S with

22. This implies that every zero of $A(z)$ belongs to $[S \cup (\gamma^* + S)]^*$.

respect to $(n-1, n)$. Then there exists a complex number γ^* such that

a. All zeros of $A(z)$ belong to $\Delta \cup (\gamma^* + \Delta)$.

b. If Δ and $\gamma^* + \Delta$ are disjoint, the number of zeros of $A(z)$ in $\gamma^* + \Delta$ is at most 1.

A slightly less precise result, contained in §§ 6—9 of Part II of [Fekete and Walsh 1957], motivated the present Theorem 16.

Proof. We set $A(z) \equiv B(z)\, D(z)$ where $D(z)$ and

$$B(z) \equiv \sum_{\nu=0}^{r} b_\nu z_\nu \qquad (b_r \neq 0)$$

are as in the proof of Theorem 11. The theorem is trivial if $r = 0$, and easily verified if $r = 1$. We thus assume that $r \geqq 2$. By the theorem following the statement of Theorem 11 (with $\overline{S}$ replacing S), $B(z)$ is a divisor of a polynomial

$$Q(z) \equiv Cg(z) + \sum_{\nu=1}^{m} \lambda_\nu \frac{g(z)}{z - z_\nu},$$

where C is a certain complex number,

$$g(z) \equiv \prod_{\nu=1}^{m} (z - z_\nu),$$

all $z_\nu \in \overline{S}$, and where λ_ν are positive numbers with

$$\sum_{\nu=1}^{m} \lambda_\nu = 1.$$

If $C = 0$, every zero of $B(z)$ belongs to $\{z_1, z_2, \dots, z_m\}^*$ which implies that all zeros of $A(z)$ belong to Δ.

In case $C \neq 0$, set $\gamma^* = -1/C$, so that

$$Q(z) \equiv C\left[g(z) - \gamma^* \sum_{\nu=1}^{m} \lambda_\nu \frac{g(z)}{z - z_\nu}\right].$$

By Walsh's Theorem stated after Theorem 14, all zeros of $B(z)$ belong to $\Delta \cup (\gamma^* + \Delta)$, and if Δ and $\gamma^* + \Delta$ are disjoint, the number of zeros of $B(z)$ in $\gamma^* + \Delta$ is at most 1. From this the theorem follows.

Theorem 17. Let S be a compact set in the z-plane, and let $A(z)\,(\not\equiv 0)$ be an n-th $(n \geqq 2)$ infrapolynomial on S with respect to $(n-1, n)$. Then there exists a complex number γ^* such that every zero ζ of $A(z)$ is of the form $c(\zeta)+\lambda(\zeta)\gamma^*$, where $c(\zeta) \in S^*$ and $0 \leqq \lambda(\zeta) \leqq 1$.[23]

Proof. We define $D(z)$ and

$$B(z) \equiv \sum_{\nu=0}^{r} b_\nu z_\nu \qquad (b_r \neq 0),$$

as in the preceding proof, and once again we can assume $r \geqq 2$. Again $B(z)$ is a divisor of a polynomial

$$Q(z) \equiv Cg(z) + \sum_{\nu=1}^{m} \lambda_\nu \frac{g(z)}{z-z_\nu},$$

where C, z_ν, $g(z)$ and λ_ν are as in the last proof. If $C=0$, every zero of $B(z)$ belongs to $\{z_1, z_2, \ldots, z_m\}^*$, and therefore every zero of $A(z)$ belongs to S^*. In case $C \neq 0$, we have

$$Q(z) \equiv C\left[g(z) - \gamma^* \sum_{\nu=1}^{m} \lambda_\nu \frac{g(z)}{z-z_\nu}\right] \qquad \left(\gamma^* = -\frac{1}{C}\right).$$

By the last lemma, every zero ζ of $B(z)$ has the form $c(\zeta)+\lambda(\zeta)\gamma^*$ $(c(\zeta) \in S^*,\ 0 \leqq \lambda(\zeta) \leqq 1)$. Since every zero ζ of $D(z)$ has obviously this form, our theorem is established. The last two theorems, as well as some of the previous ones, can be considered as generalizations of Fejér's result [Fejér, 1922, Theorems I and III].

§ 4. The location of the zeros of polynomials and rational functions of some special types.

We study here the location of the zeros of certain polynomials and rational functions, whose structure is related to those of n-th infrapolynomials considered in previous sections. Specific applications are left to the reader.

Let

$$g(z) \equiv \prod_{\nu=1}^{n} (z - z_\nu)$$

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23. Thus, $\zeta \in [S \cup (\gamma^* + S)]^*$.

be a polynomial, and let

$$f(z) \equiv g(z) - \alpha \sum_{\nu=1}^{n} \lambda_\nu \frac{g(z)}{z-z_\nu}$$

$$\left(\lambda_\nu \geqq 0\,, \quad \sum_{\nu=1}^{n} \lambda_\nu = 1\,, \quad \alpha \text{ some complex number}\right).$$

If $s = \{z_1, \ldots, z_n\}$ is a subset of a disc Γ, we know by Walsh's Theorem cited after Theorem 14, that all zeros of $f(z)$ lie in $\Gamma \cup (\alpha + \Gamma)$. We ask now whether there are sets $S\,(\supseteq s)$, more general than discs (so that s may be fitted by such an S more closely than by a disc containing s), such that all zeros of $f(z)$ lie in $S \cup (\alpha + S)$. The next theorem shows that, under a certain condition, sets $S\,(\supseteq s)$ which are the intersection of discs, have the desired property.

Theorem 18.

Hypotheses:

1. F is a non-empty family of discs, $S = \bigcap_{C \in F} C$, d is the supremum of the set of diameters of the elements of F.
2. $z_1, \ldots, z_n$ are points of S, $\lambda_1, \ldots, \lambda_n$ are non-negative reals with

$$\sum_{\nu=1}^{n} \lambda_\nu = 1\,,$$

and α is a complex number satisfying $|\alpha| > d$.

3. $g(z) \equiv \prod_{\nu=1}^{n} (z - z_\nu)$.

Conclusion: All zeros of

$$f(z) \equiv g(z) - \alpha \sum_{\nu=1}^{n} \lambda_\nu \frac{g(z)}{z-z_\nu}$$

belong to $S \cup (\alpha + S)$. The number of zeros of $f(z)$ in S and in $\alpha + S$ are, respectively, $n-1$ and 1.

Remark. In case F contains exactly one disc, Theorem 18 follows from Walsh's Theorem just mentioned.

Proof of Theorem 18. We first observe the following: let z_0 be a point of S, and z a point of some $\alpha + C$ $(C \in F)$. Then

$$|z - z_0| \geqq |\alpha| - d. \tag{24}$$

Indeed, since $z_0 \in C$ and $z - \alpha \in C$, $d \geqq |z_0 - (z - \alpha)| \geqq |\alpha| - |z - z_0|$. In particular it follows that S and $\alpha + S$ are disjoint.

Similarly, (24) holds whenever z_0 is a point of $\alpha + S$ and z is a point of some $C\,(\in F)$.

We choose an arbitrary point ζ of S. For every real t, $0 \leqq t \leqq 1$, we define $g_t(z)$, $f_t(z)$, $N_1(t)$, $N_2(t)$ and $N_3(t)$ as follows:

$$g_t(z) \equiv \prod_{\nu=1}^{n} [z - \{\zeta + t(z_\nu - \zeta)\}],$$

$$f_t(z) \equiv g_t(z) - \alpha \sum_{\nu=1}^{n} \lambda_\nu \frac{g_t(z)}{z - \{\zeta + t(z_\nu - \zeta)\}},$$

$N_1(t)$, $N_2(t)$ and $N_3(t)$ are, respectively, the numbers of zeros of $f_t(z)$ in S, $\alpha + S$, and in the complement of $S \cup (\alpha + S)$ with respect to the z-plane.

Since $f_0(z) \equiv (z - \zeta)^{n-1} [z - (\alpha + \zeta)]$, and since S and $\alpha + S$ are disjoint, $N_1(0) = n - 1$, $N_2(0) = 1$.

Since $f(z) \equiv f_1(z)$, our theorem will be proved if we can show that $N_1(1) = N_1(0)$, $N_2(1) = N_2(0)$. For this purpose it is enough to show that $N_1(t)$ and $N_2(t)$ are continuous in the interval $0 \leqq t \leqq 1$.

Let t_0 be a point of that interval. We shall show the existence of a positive δ such that if $0 \leqq t \leqq 1$ and $|t - t_0| < \delta$, then

$$N_1(t) = N_1(t_0), \quad N_2(t) = N_2(t_0). \tag{25}$$

Set

$$f_{t_0}(z) \equiv \prod_{\nu=1}^{p} (z - \zeta_\nu)^{k_\nu} \tag{26}$$

(k_ν natural numbers, $\zeta_\alpha \neq \zeta_\beta$ whenever $\alpha \neq \beta$).

Let r_ν $(\nu = 1, 2, \ldots, p)$ be positive reals such that

(27) (i) $$r_\nu \leqq |\alpha| - d,$$

(27a) (ii) if $p > 1, \ r_\nu \leqq \frac{1}{2} \min\limits_{\substack{1 \leqq j \leqq p \\ j \neq \nu}} |\zeta_\nu - \zeta_j|$

(thus, the circles $|z - \zeta_\alpha| < r_\alpha$ are mutually disjoint),

(27b) (iii) if $\zeta_\nu \notin S \cup (\alpha + S)$, r_ν is $\leqq$ the distance of ζ_ν from $S \cup (\alpha + S)$.

Let δ be a positive number such that if $0 \leqq t \leqq 1$ and $|t - t_0| < \delta$, the number of zeros of $f_t(z)$ in each of the circles $|z - \zeta_\nu| < r_\nu$ is k_ν.

Let t be a number satisfying $0 \leqq t \leqq 1$, $|t - t_0| < \delta$. We shall prove (25).

Since $N_1(\tau) + N_2(\tau) + N_3(\tau) = n$ for every τ, $0 \leqq \tau \leqq 1$, the desired equalities (25) follow from the inequalities

(27c) $N_1(t) \geqq N_1(t_0)$, $N_2(t) \geqq N_2(t_0)$, $N_3(t) \geqq N_3(t_0)$

which we proceed now to prove.

Let ζ_j be any one of $\zeta_1 \ldots \zeta_p$. We assert:

a. If $\zeta_j \in S$, all zeros of $f_t(z)$ lying in $|z - \zeta_j| < r_j$ belong to S.

b. If $\zeta_j \in \alpha + S$, all zeros of $f_t(z)$ lying in $|z - \zeta_j| < r_j$ belong to $\alpha + S$.

c. If ζ_j belongs to the complement T of $S \cup (\alpha + S)$ with respect to the z-plane, all zeros of $f_t(z)$ lying in $|z - \zeta_j| < r_j$ belong to T.

Using the defining property of δ and (27a), we infer from a, b and c the relations (27c).

Now, c follows from (27b). We prove a.

Let $\zeta_j \in S$, let $f_t(\zeta^*) = 0$, $|\zeta^* - \zeta_j| < r_j$, and let C be an element of F. We shall show that $\zeta^* \in C$, and this will prove a. By the convexity of C, all zeros of $g_t(z)$ lie in C, and therefore, by Walsh's Theorem following Theorem 14, ζ^* belongs to $C \cup (\alpha + C)$. If ζ^* belonged to $\alpha + C$, then by (24) and (27), $|\alpha| - d \leqq |\zeta^* - \zeta_j| < r_j \leqq |\alpha| - d$. Therefore $\zeta^* \in C$.

The proof of b is entirely analogous to that of a. Theorem 18 follows.

In the special case where all the λ_ν of Theorem 18 are equal, $f(z)$ is a linear combination of $g(z)$ and $g'(z)$. We can get a more general theorem, dealing with a linear combination of $g(z)$, $g'(z), \ldots, g^{(n)}(z)$. We shall state this theorem (Theorem 19) without proving it, as the proof is quite analogous to that of Theorem 18. In fact, the proof of Theorem 19

makes use of part A of the following theorem [Walsh 1922, Theorem VI] in a way similar to that in which the proof of Theorem 18 made use of the first conclusion of Walsh's Theorem mentioned there.

Theorem. Let

$$A_0, A_1, \ldots, A_n \left(n \geqq 1, \sum_{\nu=0}^{n-1} |A_\nu| > 0\right)$$

be given complex numbers, and let

$$g(z) \equiv \prod_{\nu=1}^{n} (z - z_\nu)$$

be a polynomial all of whose zeros lie in a disc $C: |z-\alpha| \leqq r$. Denote

$$f(z) \equiv A_0 g(z) + A_1 g'(z) + \ldots \qquad \ldots + A_n g^{(n)}(z),$$
$$\varphi(z) \equiv A_0 z^n + nA_1 z^{n-1} + n(n-1) A_2 z^{n-2} + \ldots + n!\, A_n,$$

and consider the discs having the zeros of $\varphi(z-\alpha)$ as centers and r as radii.

A. Every zero of $f(z)$ belongs to (at least) one of these discs.

B. If such a disc $D: |z-\delta| \leqq r$ is disjoint from all other ones, then the number of zeros of $f(z)$ in D equals the multiplicity of δ as a zero of $\varphi(z-\alpha)$.

An easy consequence of the last theorem is the following

Corollary.

A. Every zero of $f(z)$ belongs to $W+C$ (the set of all $w+c$, $w \in W$, $c \in C$) where W is the set of zeros of $\varphi(z)$.

B. If $w_0 \in W$ and if $w_0 + C$ is disjoint from every $w + C$ ($w \in W$, $w \neq w_0$), then the number of zeros of $f(z)$ in $w_0 + C$ equals the multiplicity of w_0 as a zero of $\varphi(z)$.

Theorem 19.

Hypotheses.

1. F is a non-empty family of discs, $S = \bigcap_{C \in F} C$, d is the supremum of the set of diameters of the elements of F.

2, $A_0, A_1, \dots, A_n$ $(n \geqq 1)$ are given complex numbers, $\sum_{\nu=0}^{n-1} |A_\nu| > 0$. $g(z) \equiv \prod_{\nu=1}^{n} (z-z_\nu)$ is a polynomial all of whose zeros lie in S. We denote

$$f(z) \equiv A_0 g(z) + A_1 g'(z) + \dots + A_n g^{(n)}(z).$$

$$\varphi(z) \equiv A_0 z^n + n A_1 z^{n-1} + n(n-1) A_2 z^{n-2} + \dots + n!\, A_n.$$

Conclusions: Let $\{w_1, \dots, w_p\}$ $(w_\alpha \neq w_\beta$ for $\alpha \neq \beta)$ be the set of zeros of $\varphi(z)$.

A. If a certain w_j is such that $|w_\nu - w_j| > d$ (which implies that $(w_\nu + S) \cap (w_j + S) = \emptyset$) for every ν among $1, 2, \dots, p$ which is $\neq j$, then the number of zeros of $f(z)$ in $w_j + S$ equals the multiplicity of w_j as a zero of $\varphi(z)$. In particular:

B. Suppose that $|w_\alpha - w_\beta| > d$ whenever $1 \leqq \alpha < \beta \leqq p$. Then the number of zeros of $f(z)$ in each $w_\nu + S$ $(\nu = 1, 2, \dots, p)$ equals the multiplicity of w_ν as a zero of $\varphi(z)$. Also $\{w_1 + S, \dots, w_p + S\}$ is the set of components of $\Sigma = \bigcup_{\nu=1}^{p} (w_\nu + S)$. All zeros of $f(z)$ lie in Σ.

The method of proof of Theorems 18 and 19 yields also

Theorem 20. Let S be a disc: $|z - \alpha| \leqq r$, let hypothesis 2 of Theorem 19 hold, let W denote the set of zeros of $\varphi(z)$, and let W_0 be a subset of W.

A. If $W_0 + S$ and $(W - W_0) + S$ are disjoint, then the number of zeros of $f(z)$ in $W_0 + S$ equals the number of zeros of $\varphi(z)$ in W_0. Slightly differently formulated:

B. If $|w_1 - w_0| > 2r$ for every w_0 belonging to W_0 and every w_1 belonging to $W - W_0$, then the number of zeros of $f(z)$ in $W_0 + S$ equals the number of zeros of $\varphi(z)$ in W_0.

We turn now to the study of the zeros of functions of the form

$$\sum_{\nu=0}^{p} A_\nu z^\nu + \sum_{\nu=1}^{q} B_\nu z^{-\nu} + \sum_{\nu=1}^{n} \lambda_\nu \frac{1}{z-z_\nu} \tag{28}$$

$$(A_\nu\,,\ B_\nu \text{ real},\ \lambda_\nu > 0),$$

under certain conditions, and obtain results related to Jensen's Theorem [Jensen 1913. Proof in Walsh 1920].

The expression (28) includes, as special cases, many of the functions encountered in the preceding pages.

Theorem 21. Let $\lambda_1, \ldots, \lambda_n$ be positive, A_0 real, $A_1, A_2, \ldots, A_p$ non-positive reals, and $B_1, B_2, \ldots, B_q$ non-negative reals. Let $z_1, z_2, \ldots, z_n$ be arbitrary points of the z-plane, and let

$$f(z) \equiv \sum_{\nu=0}^{p} A_\nu z^\nu + \sum_{\nu=1}^{q} B_\nu z^{-\nu} + \sum_{\nu=1}^{n} \lambda_\nu \left(\frac{1}{z-z_\nu} + \frac{1}{z-\bar{z}_\nu} \right).$$

Then every non-real zero z_0 of $f(z)$, satisfying

$$|\arg z_0| \leqq \min\left(\frac{\pi}{p}\,,\ \frac{\pi}{q}\right),\ ^{(24)}$$

belongs to at least one of the discs

$$\left| z - \frac{z_\nu + \bar{z}_\nu}{2} \right| \leqq \frac{1}{2} |z_\nu - \bar{z}_\nu|$$

(in particular, in case $p = q = 1$, every non-real zero of $f(z)$ belongs to at least one of these discs).

The proof relies on the following Lemma [Walsh 1920].

Lemma. Let ζ and z_0 be points in the z-plane and let $\operatorname{Im}(z_0)$ be $\neq 0$. If z_0 does not belong to the disc

$$\left| z - \frac{\zeta + \bar{\zeta}}{2} \right| \leqq \frac{1}{2} |\zeta - \bar{\zeta}|,$$

then

$$\operatorname{Im}\left(\frac{1}{z_0 - \zeta} + \frac{1}{z_0 - \bar{\zeta}} \right)$$

24. arg denotes the principal value of the argument.

is not zero and its sign is that of $\operatorname{Im}(z_0)$. [See the last mentioned paper for the physical interpretation of this result].

Proof of Theorem 21. Let z_0 be a non-real zero of $f(z)$, let

$$|\arg z_0| \leqq \min\left(\frac{\pi}{p}, \frac{\pi}{q}\right)$$

and suppose that z_0 does not belong to any one of the discs mentioned in the Theorem.

By the last Lemma (with $\zeta = z_\nu$)

$$\operatorname{Im}(z_0)\operatorname{Im}\left(\frac{1}{\overline{z_0 - z_\nu}} + \frac{1}{\overline{z_0 - \overline{z_\nu}}}\right) > 0 \quad (\nu = 1, 2, \ldots, n).$$

Our assumptions about z_0 imply also that:

$$\operatorname{Im}(z_0)\operatorname{Im}(\overline{z_0^\nu}) \leqq 0 \qquad (\nu = 1, 2, \ldots, p)$$

$$\operatorname{Im}(z_0)\operatorname{Im}(\overline{z_0^{-\nu}}) \geqq 0 \qquad (\nu = 1, 2, \ldots, q).$$

Therefore

$$0 = \operatorname{Im}(z_0)\operatorname{Im}(\overline{f(z_0)}) = \sum_{\nu=0}^{p} A_\nu \operatorname{Im}(z_0)\operatorname{Im}(\overline{z_0^\nu})$$

$$+ \sum_{\nu=1}^{q} B_\nu \operatorname{Im}(z_0)\operatorname{Im}(\overline{z_0^{-\nu}}) + \sum_{\nu=1}^{n} \lambda_\nu \operatorname{Im}(z_0)\operatorname{Im}\left(\frac{1}{\overline{z_0 - z_\nu}} + \frac{1}{\overline{z_0 - \overline{z_\nu}}}\right) > 0,$$

a contradiction. This proves the theorem.

Theorem 22. Let $\lambda_1, \ldots, \lambda_n$ be positive, $A_0, \ldots, A_p$ non-positive reals, and $B_1, \ldots, B_q$ $(q \geqq 2)$ non-negative reals. Let $z_1, \ldots, z_n$ be points of the half-plane $\operatorname{Re}(z) > 0$, and let $f(z)$ be defined as in Theorem 21.

Let z_0 be a non-real zero of $f(z)$ satisfying

$$|\arg z_0| \leqq \min\left(\frac{\pi}{p+1}, \frac{\pi}{q-1}\right).$$

Then there exists a ν, $1 \leqq \nu \leqq n$ such that $\operatorname{Im}(z_\nu) \neq 0$, and such that z_0 belongs to the closed interior of the circle passing through z_ν and $\overline{z_\nu}$, and tangent to the line Oz_ν.

Similarly to the proof of Theorem 21, we rely here on the following [cf. Walsh 1955]

Lemma. Let ζ and z_0 be non-real points of the z-plane, and let $\alpha\,(\neq \mathrm{Re}(\zeta))$ be real. Let M be the circle passing through ζ and $\bar{\zeta}$ and tangent to the line $\alpha\zeta$. Suppose that z_0 lies exterior to M.

Let a, b be reals defined by

$$\frac{1}{z_0-\zeta}+\frac{1}{z_0-\bar{\zeta}}=a\,(z_0-\alpha)+ib\,(z_0-\alpha)\,.$$

Then $b\neq 0$, and the sign of b is that of $\mathrm{Im}(z_0)\,[\mathrm{Re}(\zeta)-\alpha]$.

Proof of Theorem 22. Suppose there exists no ν as claimed in the theorem.

Let ν be any one of $1, 2, \ldots, n$. If $\mathrm{Im}(z_\nu)\neq 0$, then by the last Lemma (with $\zeta=z_\nu$, $\alpha=0$)

$$\mathrm{Im}(z_0)\,\mathrm{Im}\left[z_0^{-1}\left(\frac{1}{z_0-z_\nu}+\frac{1}{z_0-\bar{z}_\nu}\right)\right]>0\,.$$

This relation is easily seen to hold also if $\mathrm{Im}(z_\nu)=0$.

For $\nu=0, 1, \ldots, p$, $\mathrm{Im}(z_0^{-1}\,\overline{z_0^{\nu}})=|z_0|^{2\nu}\,\mathrm{Im}(z_0^{-(\nu+1)})$, and the right hand side, if not zero, has a sign opposite to that of $\mathrm{Im}(z_0)$, so that

$$\mathrm{Im}(z_0)\,\mathrm{Im}(z_0^{-1}\,\overline{z_0^{\nu}})\leqq 0\,.$$

Similarly, for $\nu=1, 2, \ldots, q$, $\mathrm{Im}(z_0^{-1}\,\overline{z_0^{-\nu}})$, if $\neq 0$, has the sign of $\mathrm{Im}(z_0)$, so that

$$\mathrm{Im}(z_0)\,\mathrm{Im}(z_0^{-1}\,\overline{z_0^{-\nu}})\geqq 0\,.$$

From the last three inequalities and from our hypotheses on the A_ν and the B_ν, we get:

$$0=\mathrm{Im}(z_0)\,\mathrm{Im}(z_0^{-1}\,\overline{f(z_0)})=\sum_{\nu=0}^{p}A_\nu\,\mathrm{Im}(z_0)\,\mathrm{Im}(z_0^{-1}\,\overline{z_0^{\nu}})$$

$$+\sum_{\nu=1}^{q}B_\nu\,\mathrm{Im}(z_0)\,\mathrm{Im}(z_0^{-1}\,\overline{z_0^{-\nu}})+\sum_{\nu=1}^{n}\lambda_\nu\,\mathrm{Im}(z_0)\,\mathrm{Im}\left[z_0^{-1}\left(\frac{1}{z_0-z_\nu}+\frac{1}{z_0-\bar{z}_\nu}\right)\right]>0\,,$$

a contradiction. This establishes the theorem.

BIBLIOGRAPHY

L Fejér.

1922. Über die Lage der Nullstellen von Polynomen, die aus Minimumforderungen gewisser Art entspringen, *Math. Ann.*, vol. 85, pp 41—48.

M. Fekete.

1951. On the structure of extremal polynomials, *Proc. Nat. Acad. Sciences*, vol. 37, pp. 95—103.

1955. On the structure of polynomials of least deviation, *Bull. Res. Council of Israel*, vol. 5A, pp 11—19.

M. Fekete and J. L. von Neumann.

1922. Über die Lage der Nullstellen gewisser Minimumpolynome, *Jahresber. Deut. Math.-Verein.* vol. 31, pp. 125—138.

M. Fekete and J. L. Walsh.

1955. On the asymptotic behavior of polynomials with extremal properties, and of their zeros, *J. d'Analyse Math.*, vol. 4, pp. 49—87.

1957. On restricted infrapolynomials, Ibid., vol. 5, pp. 47—76.

1957a. Asymptotic behavior of restricted extremal polynomials and of their zeros, *Pacific J. of Math.*, vol. 7, pp. 1037—1064.

J. L. W. Jensen.

1913. Recherches sur la théorie des équations, *Acta Math.*, vol. 36, pp. 181—185.

E. Laguerre.

1898. Oeuvres, vol. 1, p. 48, Paris.

M. Marden.

1949. The geometry of the zeros of a polynomial in a complex variable, Mathematical Surveys (Amer. Math. Soc.), No. III, New York.

1959. On the zeros of infrapolynomials for partly arbitrary point sets, *Proc. Amer. Math. Soc.*, vol. 10, pp. 391—394.

T. S. Motzkin and J. L. Walsh.

1953. On the derivative of a polynomial and Chebyshev approximation, *Proc. Amer. Math. Soc.* vol. 4, pp. 76—87.

1955. Least pth power polynomials on a real finite point set, *Trans. Amer. Math. Soc.*, vol. 78, pp. 67—81.

1956. Least pth power polynomials on a finite point set, Ibid., vol. 83, pp. 371—396.

1957. Polynomials of best approximation on a real finite point set, *Proc. Nat. Acad. Sciences*, vol. 43, pp. 845—846.

1957a. Underpolynomials and infrapolynomials, *Illinois J. of Math.*, vol. 1, pp 406—426.

1959. Location of zeros of infrapolynomials, *Compositio Math.*, vol. 14, fasc. 1, pp. 50—70.

O. Shisha.

1956. Extremal systems in the Euclidean n-spaces E_n (abstract), *Bull. Res. Council of Israel*, vol. 5A. p. 198.

1957. On the location of the zeros of infrapolynomials (abstract), Ibid., vol. 7F, pp. 44—45.

1958. Properties of extremal systems of points, Ph. D. thesis, The Hebrew University, Jerusalem (mimeographed).

1958a. Nearest functions (abstract), Notices Amer. Math. Soc., vol. 5, No. 7, p. 853.

1959. A characterization of certain nearest functions (abstract), Idid., vol. 6, No. 2. p. 144.

1959a. On generalized infrapolynomials of a certain type (abstract), Ibid., vol. 6, No. 3. p. 283.
1959b. On the structure of infrapolynomials with prescribed first and last coefficients (abstract), Ibid., vol. 6. No. 4, p. 393.
1961. A structure theorem for infrapolynomials with prescribed coefficients (abstract), Ibid.. vol. 8, No. 1. p. 66.
T. Takagi.
1921. Note on the algebraic equations, *Proc. Physics-math. Soc. Japan*, vol. 3, pp. 175—179.
J.L. Walsh.
1920. On the location of the roots of the derivative of a rational function, *Ann, of Math*. vol. 22, pp. 128—144.
1922. On the location of the roots of certain types of polynomials, *Trans. Amer. Math. Soc.*, vol. 24, pp. 163—180.
1924. On the location of the roots of polynomials, *Bull. Amer. Math. Soc.*, vol. 30, pp. 51—62.
1950. The Location of Critical Points. *Amer. Math. Soc. Colloquium Pub.* vol. 34.
1955. A generalization of Jensen's theorem on the zeros of the derivative of a polynomial, *Amer. Math. Monthly*, vol, 62, pp. 91—93.
1958. On infrapolynomials with prescribed constant term. *J. Math. Pures Appl.* vol. 37, pp. 295—316.
1959. On extremal approximation, On numerical approximation, edited by R. E. Langer, The University of Wisconsin Press, Madison, pp. 209—216.
J.L. Walsh and T. S. Motzkin.
1957. Polynomials of best approximation on a real finite point set, *Proc. Nat. Acad. Sciences*, vol. 43, pp. 845—846.
1959. Polynomials of best approximation on an interval, Ibid,, vol. 45, pp. 1523—1528.
J.L. Walsh and M. Zedek.
1956. On generalized Tchebycheff polynomials, Ibid, vol. 42, pp. 99—104.
M. Zedek.
1955. Fejér's theorem on the zeros of extremal polynomials generalized, preliminary report, *Bull. Amer. Math. Soc.*, vol. 61, p. 49.
1957. Infra-(n,s)-polynomials on a real set, preliminary report, Ibid., vol 63, p. 25.
1957a. Strong infra-(n,s)-polynomials, preliminary report, Ibid., vol. 63, p. 127.

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National Bureau of Standards

(Received November 4, 1960)

CURVATURE PROPERTIES OF TEICHMÜLLER'S SPACE*

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Introduction

For background we refer to the author's paper [3] and to a joint paper with L. Bers [4], to be quoted as (A) and (AB).

Without details, a point in Teichmüller's space T_g is represented by a closed Riemann surface of fixed genus $g(>1)$ together with an outer automorphism of its fundamental group. Intrinsic definitions lead to a metric on T_g, introduced by Teichmüller, to a Riemannian structure whose use was suggested by A. Weil, and finally to a complex analytic structure of dimension 3g—3. It was proved by Weil, and again in (A), with very little computation, that the Riemannian metric is Kählerian with respect to the complex structure.

It was reasonable to conjecture that the metric has negative curvature. The main purpose of this paper is to verify that the Ricci curvatures and the curvatures of holomorphic sections are indeed negative. The proof is by explicit computations which turn out to be less formidable than might have been expected.

1. Instead of using the unit disk, as in (A), we shall work with the upper halfplane $H = \{z = x + iy \mid y > 0\}$. The advantages are purely formal.

We use the familiar notations

$$\partial f = \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

$$\bar{\partial} f = \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

where $\partial f/\partial x$, $\partial f/\partial y$ are weak L^2-derivatives. Let μ be a complex-valued measurable function on H with $\|\mu\|_\infty < 1$. There exists a unique homeo-

* Work supported by the Air Force Office of Scientific Research.

morphic mapping of H onto itself, to be denoted by f^μ, which satisfies

$$\bar\partial f^\mu = \mu\, \partial f^\mu \tag{1.1}$$

and is normalized by

$$f^\mu(0) = 0\,, \quad f^\mu(1) = 1\,, \quad f^\mu(\infty) = \infty. \tag{1.2}$$

The normalization makes sense, for f^μ has a unique extension to a homeomorphism of the closed half-plane.

The mappings f^μ are quasiconformal. Conversely, any quasiconformal mapping has weak L^2-derivatives which satisfy an equation of form (1.1) with $\|\mu\|_\infty < 1$. There is thus one-to-one correspondence between normalized quasiconformal homeomorphisms and elements of $L^\infty(H)$ with norm < 1. One finds that $f^\lambda = f^\rho \circ f^\mu$ if and only if

$$\rho = \left[\frac{\lambda - \mu}{1 - \bar\mu\lambda}\,(\partial f^\mu / \overline{\partial f^\mu})\right] \circ (f^\mu)^{-1}. \tag{1.3}$$

This relation may be regarded as defining a group structure on the unit ball of $L^\infty(H)$, and we set $\lambda = \rho \cdot \mu$.

Throughout this paper the extreme generality which results from considering functions μ which are merely bounded and measurable is completely unnecessary. We shall therefore assume, henceforth, that all functions μ are real analytic. As a consequence the functions f^μ will also be real analytic. What is more, if μ depends analytically on a number of parameters, then $f^\mu(z)$ will be simultaneously analytic in z and the parameters (see (AB)).

2. Consider a function F on $L^\infty(H)$ with complex values that we denote either by F^μ or $F(\mu)$, according to convenience. We say that F is differentiable if $F(\mu + t_1 \nu_1 + \ldots + t_k \nu_k)$ is a differentiable function of the real variables $t_1, \ldots, t_k$ for all $\mu, \nu_1, \ldots, \nu_k \in L^\infty(H)$. We set

$$\dot F^\mu[\nu] = \frac{\partial}{\partial t} F(\mu + t\nu)_{\,|\, t=0} \tag{2.1}$$

and

$$\begin{aligned} DF^\mu[\nu] &= \frac{1}{2}(\dot F^\mu[\nu] - i \dot F^\mu[i\nu]) \\ \bar D F^\mu[\nu] &= \frac{1}{2}(\dot F^\mu[\nu] + i \dot F^\mu[i\nu]). \end{aligned} \tag{2.2}$$

Here $\dot{F}^{\mu}[\nu]$ is linear in ν over the reals, while $DF^{\mu}[\nu]$ is linear and $\bar{D}F^{\mu}[\nu]$ is antilinear over the complex numbers.

For simplicity, when the differentiation is at $\mu = 0$ we suppress the reference to μ in the notations. All differentiations can be reduced to this case. Indeed, define a function F_{μ} by setting $F_{\mu}(\rho) = F(\rho \cdot \mu)$. It then follows by (1.3) that

$$\dot{F}^{\mu}[\nu] = \dot{F}_{\mu}[L^{\mu}\nu] \tag{2.3}$$

where

$$L^{\mu}\nu = \left[\frac{\nu}{1-|\mu|^2}(\partial f^{\mu}/|\partial f^{\mu}|)^2\right] \circ (f^{\mu})^{-1}, \tag{2.4}$$

and similarly

$$DF^{\mu}[\nu] = DF_{\mu}[L^{\mu}\nu], \quad \bar{D}F^{\mu}[\nu] = \bar{D}F_{\mu}[L^{\mu}\nu]. \tag{2.5}$$

In particular, f^{μ} is differentiable, and because $f^{\rho}_{\mu} = f^{\rho} \circ f^{\mu}$ we obtain

$$\dot{f}^{\mu}[\nu] = \dot{f}[L^{\mu}\nu] \circ f^{\mu}. \tag{2.6}$$

The normalization implies $\dot{f}(0) = \dot{f}(1) = 0$ and $\dot{f}/z^2 \to 0$ for $z \to \infty$. Moreover, $\dot{f}[\nu]$ is real on the real axis.

It follows from (1.1) that

$$\bar{\partial}\dot{f}[\nu] = \nu \tag{2.7}$$

and hence

$$\bar{\partial} D f[\nu] = \nu, \quad \bar{\partial}\bar{D} f[\nu] = 0. \tag{2.8}$$

The last equation shows that $\bar{D}f[\nu]$ is an analytic function.

The solutions of (2.7) are of the form

$$\dot{f}[\nu](\zeta) = -\frac{1}{\pi}\int_H P(z,\zeta)\,\nu(z)\,dx\,dy + \Phi(\zeta) \tag{2.9}$$

where

$$P(z,\zeta) = \frac{1}{z-\zeta} + \frac{\zeta-1}{z} - \frac{\zeta}{z-1} \tag{2.10}$$

and Φ is analytic. Because of the boundary conditions it follows easily by use of the reflection principle that

$$\dot{f}[\nu](\zeta) = -\frac{1}{\pi}\int_H (P(z,\zeta)\,\nu(z) + P(\bar{z},\zeta)\,\nu(z))\,dx\,dy \tag{2.11}$$

and consequently

$$(2.12)\qquad \begin{aligned} Df[\nu](\zeta) &= -\frac{1}{\pi}\int_H P(z,\zeta)\,\nu(z)\,dx\,dy \\ \bar{D}f[\nu](\zeta) &= -\frac{1}{\pi}\int_H P(\bar{z},\zeta)\,\bar{\nu}(z)\,dx\,dy. \end{aligned}$$

For an arbitrary μ we obtain from here, by use of (2,6), (2.4) and a change of integration variable,

$$(2.13)\qquad \begin{aligned} Df^{\mu}[\nu](\zeta) &= -\frac{1}{\pi}\int_H P(f^{\mu}(z),f^{\mu}(\zeta))\,\partial f^{\mu}(z)^2\,\nu(z)\,dx\,dy \\ \bar{D}f^{\mu}[\nu](\zeta) &= -\frac{1}{\pi}\int_H P(\overline{f^{\mu}}(z),f^{\mu}(\zeta))\,\overline{\partial f^{\mu}}(z)^2\,\bar{\nu}(z)\,dx\,dy. \end{aligned}$$

3. Let Γ be a discontinuous group of linear transformations of H onto itself whose quotient space $W = H/\Gamma$ is a closed Riemann surface of genus $g > 1$. We say that $\mu \in L^{\infty}(H)$ is a Beltrami differential on W (or with respect to Γ) if it satisfies

$$(3.1)\qquad (\mu \circ A)\;\bar{A}'(z) = \mu A'(z)$$

for all $A \in \Gamma$. The linear space of Beltrami differentials is denoted by $B(\Gamma)$, its unit ball by $B_1(\Gamma)$. If $\mu \in B_1(\Gamma)$ we can form f^{μ}. It is found that

$$(3.2)\qquad f^{\mu} \circ A = A^{\mu} \circ f^{\mu}$$

where A^{μ} is a linear transformation. The A^{μ} form a group Γ^{μ}, and the mapping $A \to A^{\mu}$ is an isomorphism. These isomorphisms may be identified with the points of T_g. In other words, if R denotes the equivalence relation $\mu_1 \sim \mu_2$ defined by $A^{\mu_1} = A^{\mu_2}$ for all $A \in \Gamma$, when T_g can be canonically identified with the space $B_1(\Gamma)/R$.

Let $N(\Gamma)$ be the linear subspace of $B(\Gamma)$ which consists of all Beltrami differentials ν satisfying the condition $Df[\nu] = 0$. By (2.12) this means that

$$(3.3)\qquad \int_H P(z,\bar{\zeta})\,\nu(z)\,dx\,dy = 0,$$

and by (2.11) a further equivalent condition is that $\dot{f}[\nu]$ vanishes on the

real axis. One proves also that $\nu \in N(\Gamma)$ if and only if

$$\int_{H/\Gamma} \varphi(z)\, \nu(z)\, dx\, dy = 0 \tag{3.4}$$

for all quadratic differentials φ. We recall that φ is called a quadratic differential on W if it is analytic and satisfies

$$(\varphi \circ A)\, A'(z)^2 = \varphi \tag{3.5}$$

for all $A \in \Gamma$.

4. It is worth while to give a brief proof of the equivalence of (3.3) and (3.4). Although it is essentially the same proof as in (A), the form we give it serves to emphasize some points which will be needed in the sequel.

We introduce the notation

$$K(z\zeta) = (z - \zeta)^{-2} \tag{4,1}$$

and observe that

$$\partial^3 P / \partial \zeta^3 = 6K(z\zeta)^2 . \tag{4.2}$$

One verifies that

$$K(Az\, A\zeta)\, A'(z)\, A'(\zeta) = K(z\zeta) \tag{4.3}$$

for all linear transformations A, and that

$$K(Az\, \overline{A}\zeta)\, A'(z)\, \overline{A}'(\zeta) = K(z\overline{\zeta}) \tag{4.4}$$

for linear transformations which leave the real axis invariant.

We collect in a lemma the following information:

Lemma 1.

a) For $\nu \in B(\Gamma)$ the formula

$$\varphi(\zeta) = \frac{12}{\pi} \int_H K(\overline{z\zeta})^2\, \overline{\nu}(z)\, dx\, dy \tag{4.5}$$

defines a quadratic differential $\varphi = \varphi[\nu]$.

b) If φ is a quadratic differential, then $\nu = \overline{\varphi} y^2$ is in $B(\Gamma)$, and $\varphi[\nu] = \varphi$.

c) If $\nu_1, \nu_2 \in B(\Gamma)$, then

$$\int_{H/\Gamma} \nu_1\, \varphi[\nu_2]\, dx\, dy = \int_{H/\Gamma} \overline{\nu_2}\, \overline{\varphi}[\nu_1]\, dx\, dy . \tag{4.6}$$

Part a) follows by use of (4.4). Part b) amounts to a Bergman type reproduction formula

$$\varphi(\zeta) = \frac{12}{\pi} \int_H K(\bar{z}\zeta)^2 \varphi(z)\, y^2\, dx\, dy\,, \tag{4.7}$$

which is easily proved by means of Stokes' formula and residues. To prove c), let H/Γ be represented by a "fundamental region" Δ. In order to exhibit the complex integration variable we shall henceforth denote the area element by $d\sigma(z)$. By utilization of (4.4) we obtain

$$\int_{H/\Gamma} \nu_1 \varphi[\nu_2]\, d\sigma(z) =$$

$$= \int_\Delta \nu_1(z)\, d\sigma(z) \int_H K(\bar{\zeta}z)^2 \overline{\nu_2}(\zeta)\, d\sigma(\zeta)$$

$$= \sum_{A \in \Gamma} \int_\Delta \nu_1(z)\, d\sigma(z) \int_{A\Delta} K(\bar{\zeta}z)^2 \overline{\nu_2}(\zeta)\, d\sigma(\zeta)$$

$$= \sum_{A \in \Gamma} \int_{A^{-1}\Delta} \nu_1(Az)\, d\sigma(Az) \int_\Delta K(\overline{A\zeta}\, Az)^2 \overline{\nu_2}(A\zeta)\, d\sigma(A\zeta)$$

$$= \sum_{A \in \Gamma} \int_{A^{-1}\Delta} \nu_1(z)\, d\sigma(z) \int_\Delta K(\bar{\zeta}z)^2 \overline{\nu_2}(\zeta)\, d\sigma(\zeta)$$

$$= \int_H \nu_1(z)\, d\sigma(z) \int_\Delta K(\bar{\zeta}z)\, \overline{\nu_2}(\zeta)\, d\sigma(\zeta)$$

$$= \int_{H/\Gamma} \overline{\nu_2}\, \overline{\varphi}[\nu_1]\, d\sigma(\zeta)\,.$$

We remark that (2.12), (4.2) and (4.5) imply

$$\partial^3 \bar{D} f[\nu] = -\frac{1}{2} \varphi[\nu]\,. \tag{4.8}$$

In other words, the third derivative of the analytic function $\bar{D}f$ is always a quadratic differential.

Suppose now that ν belongs to $N(\Gamma)$. Then (4.8) shows that $\varphi[\nu] = 0$. By Lemma 1b any quadratic differential φ_0 is of the form $\varphi[\nu_0]$, and c) gives

$$\int_{H/\Gamma} \varphi_0 \nu \, d\sigma = \int_{H/\Gamma} \overline{\nu_0} \overline{\varphi} [\nu] \, d\sigma = 0 .$$

We have thus shown that (3.3) implies (3.4). Conversely, if (3.4) holds we find by c) that

$$\int_{H/\Gamma} \nu_0 \varphi [\nu] \, d\sigma = 0$$

for all ν_0, and in particular for $\nu_0 = \varphi [\nu] y^2$. Hence $\varphi [\nu]$ must vanish identically. But then $\overline{D} f [\nu]$ is a polynomial of at most second degree, and the normalization shows that it must reduce to 0. We conclude that $\nu \in N(\Gamma)$.

5. If we write

$$(5.1) \qquad \langle \nu_1 , \nu_2 \rangle = \int_{H/\Gamma} \nu_1 \varphi [\nu_2] \, d\sigma$$

it follows by c) that $\langle \nu_1 , \nu_2 \rangle = \overline{\langle \nu_2 , \nu_1 \rangle}$. The Hermitian character appears more clearly if we use the fact that $\nu - \overline{\varphi} [\nu] y^2 \in N(\Gamma)$, or as we shall also write, $\nu \equiv \overline{\varphi} [\nu] y^2 \mod N(\Gamma)$. We conclude that

$$(5.2) \qquad \langle \nu_1 , \nu_2 \rangle = \int_{H/\Gamma} \overline{\varphi} [\nu_1] \varphi [\nu_2] y^2 \, dx \, dy$$

and in particular

$$(5.3) \qquad \langle \nu , \nu \rangle = \int_{H/\Gamma} |\varphi [\nu]|^2 y^2 \, dx \, dy .$$

The latter relation shows that $\langle \nu , \nu \rangle = 0$ if and only if $\nu \in N(\Gamma)$. Hence the inner product $\langle \nu_1 , \nu_2 \rangle$ defines a Hermitian metric on the quotient space $B(\Gamma) / N(\Gamma)$. Because there are $n = 3g - 3$ linearly independent quadratic differentials this space has the complex dimension n.

6. We choose now a relative basis $\nu_1 , \dots , \nu_n$ of $B(\Gamma)$ modulo $N(\Gamma)$. If $t = (t_1 , \dots , t_n)$ denotes a complex vector we set

$$|t| = (|t_1|^2 + \dots + |t_n|^2)^{1/2}$$

and

$$\nu(t) = t_1 \nu_1 + \ldots + t_n \nu_n .$$

By elementary reasoning (see (A)) one proves the following:

1) There exists $\varepsilon > 0$ such that if $|t|$, $|t'| < \varepsilon$, then $\nu(t)$ and $\nu(t')$ are equivalent in the sense that $A^{\nu(t)} = A^{\nu(t')}$ for all $A \in \Gamma$ only if $t = t'$.

2) There exists $\delta > 0$ such that every $\mu \in B(\Gamma)$ with $\|\mu\|_\infty < \delta$ is equivalent to a $\nu(t)$, $|t| < \varepsilon$.

We conclude from this result that T_g has a manifold structure, and that $t_1, \ldots, t_n$ serve as local coordinates in a neighborhood of the initial point H/Γ. Indeed, the construction can be repeated for an arbitrary initial point H/Γ^μ, and it is easy to show that overlapping coordinates are connected by bicontinuous relations.

What is more, the coordinates $t_1, \ldots, t_n$ define a complex structure on T_g.[1] To see this, we observe that a function $F(\mu)$, defined near $\mu = 0$, is a function on T_g if it has equal values at equivalent points. The coordinate functions $t_k(\mu)$ have this property, and one shows (see (A)) that they are complex analytic functions of μ in the sense that $\bar{D}t_k(\mu)[\nu] = 0$ for all $\nu \in B(\Gamma)$. It follows easily that $F(\mu)$ is analytic in μ if and only if $F(\nu(t))$ depends analytically on $t_1, \ldots, t_n$. As in **2**, we introduce the function F_μ with values $F_\mu(\rho) = F(\rho \cdot \mu)$. As a consequence of (2.3), F and F_μ are simultaneously analytic, and one concludes that the coordinates $t_1^\mu, \ldots, t_n^\mu$ which correspond to the basis elements $L^\mu \nu_1, \ldots, L^\mu \nu_n$ at the point H/Γ^μ depend analytically on $t_1, \ldots, t_n$.

With respect to the complex structure the tangent space at the origin is spanned by the derivations

$$\partial F / \partial t_\alpha = DF[\nu_\alpha] \tag{6.1}$$

and there is thus a natural identification of the tangent space with $B(\Gamma)/N(\Gamma)$. To define a Riemannian structure we must introduce a Hermitian structure on the tangent space, and it is natural to do so by use of the inner product $\langle \nu_1, \nu_2 \rangle$ defined by (5.1). More generally, the derivations at a point $\mu = \nu(t)$ are

$$\partial F / \partial t_\alpha = DF^\mu[\nu_\alpha] = DF_\mu[L^\mu \nu_\alpha] \tag{6.2}$$

1. This was first proved by L. Bers.

and we are thus led to set $ds^2 = \Sigma\, g_{\alpha\overline{\beta}}(t)\, dt_\alpha\, d\overline{t_\beta}$ with

$$g_{\alpha\overline{\beta}}(t) = \langle L^\mu \nu_\alpha\,,\, L^\mu \nu_\beta \rangle\,, \tag{6.3}$$

where we remind the reader that $\mu = t_1 \nu_1 + \ldots + t_n \nu_n$.

Explicitly, (5.1) and (4,5) give

$$g_{\alpha\overline{\beta}}(t) = \frac{12}{\pi} \int_{H/\Gamma^\mu} \int_H K(u\overline{v})^2\, L^\mu \nu_\alpha(u)\, \overline{L}^\mu \nu_\beta(v)\, d\sigma(u)\, d\sigma(v)\,. \tag{6.4}$$

Formula (2.4) suggests changing the integration variables, and on introducing the notation

$$K^\mu(u\overline{v}) = K(f^\mu(u)\, \overline{f^\mu(v)})\, \partial f^\mu(u)\, \overline{\partial f^\mu(v)} \tag{6.5}$$

we find that (6.4) can be rewritten in the form

$$g_{\alpha\overline{\beta}}(t) = \frac{12}{\pi} \int_{H/\Gamma} \int_H K^\mu(u\overline{v})^2\, \nu_\alpha(u)\, \overline{\nu_\beta}(v)\, d\sigma(u)\, d\sigma(v)\,. \tag{6.6}$$

In view of the Hermitian symmetry we may also use the alternate formula

$$g_{\alpha\overline{\beta}}(t) = \frac{12}{\pi} \int_{H/\Gamma} \int_H K^\mu(\overline{uv})^2\, \overline{\nu_\beta}(u)\, \nu_\alpha(v)\, d\sigma(u)\, d\sigma(v)\,. \tag{6.7}$$

To obtain a third formula we use (5.2) instead of (5.1). We find that

$$g_{\alpha\overline{\beta}} = \frac{144}{\pi^2} \int_{H/\Gamma} \int_H \int_H K(u\overline{z})^2\, K(v\overline{z})^2\, y^2\, L^\mu \nu_\alpha(u)\, \overline{L}^\mu \nu_\beta(v)\, d\sigma(z)\, d\sigma(u)\, d\sigma(v) \tag{6.8}$$

and when the variables are changed this becomes

$$g_{\alpha\overline{\beta}}(t) = \tag{6.9}$$

$$= -\frac{36}{\pi^2} \int_{H/\Gamma} \int_H \int_H \frac{K^\mu(u\overline{z})^2\, K^\mu(\overline{v z})^2}{K^\mu(z\overline{z})} (1 - |\mu(z)|^2)\, \nu_\alpha(u)\, \overline{\nu_\beta}(v)\, d\sigma(z)\, d\sigma(u)\, d\sigma(v)\,.$$

One proves easily, for instance by use of Lemma 1c, that the right hand side of (6.9) does not change if we let any one of the integration variables range over H/Γ, while the others range over H. This remark applies to all future formulas of the same kind.

7. Our next task is to compute the partial derivatives of $g_{\alpha\overline{\beta}}$. As we have pointed out before, we lose no generality by assuming that the ν_k are real analytic. It will therefore be legitimate to interchange

the orders of derivations and integrations as soon as all integrals are uniformly convergent.

In analogy with (6.5) we shall write

$$K^{\mu}(uv) = K(f^{\mu}(u)\, f^{\mu}(v))\, \partial f^{\mu}(u)\, \partial f^{\mu}(v) \tag{7.1}$$

and we observe that

$$\begin{aligned} K^{\mu}(uv) &= \frac{\partial^2}{\partial u\, \partial v} \log (f^{\mu}(u) - f^{\mu}(v)) \\ K^{\mu}(u\bar{v}) &= \frac{\partial^2}{\partial u\, \partial \bar{v}} \log (f^{\mu}(u) - \overline{f^{\mu}}(v)) . \end{aligned} \tag{7.2}$$

To determine the derivatives of K^{μ} we start from formula (2.13), and obtain

$$\frac{\partial}{\partial t_{\gamma}} \log (\overline{f^{\mu}}(u) - f^{\mu}(v)) =$$

$$= -\frac{1}{\pi} \int_H \frac{P(f^{\mu}(z)\, ,\, \overline{f^{\mu}}(u)) - P(f^{\mu}(z)\, ,\, f^{\mu}(v))}{\overline{f^{\mu}}(u) - f^{\mu}(v)}\, \partial f^{\mu}(z)^2\, \gamma_{\gamma}(z)\, d\sigma(z) .$$

Differentiation with respect to $\bar{u}$ offers no difficulty, and it is a standard result that we may also differentiate with respect to v, provided that the resulting integral is interpreted as a principal value, We find in this manner

$$\frac{\partial}{\partial t_{\gamma}} K^{\mu}(\bar{u}v) = -\frac{1}{\pi} \int_H K^{\mu}(z\bar{u})\, K^{\mu}(zv)\, \nu_{\gamma}(z)\, d\sigma(z) , \tag{7.3}$$

and by a similar calculation

$$\frac{\partial}{\partial t_{\gamma}} K^{\mu}(uv) = -\frac{1}{\pi} \int_H K^{\mu}(zu)\, K^{\mu}(zv)\, \gamma_{\gamma}(z)\, d\sigma(z) . \tag{7.4}$$

With the help of this result (6.7) yields, at least formally

$$\frac{\partial}{\partial t_{\gamma}} g_{\alpha\bar{\beta}} = \tag{7.5}$$

$$= -\frac{24}{\pi^2} \int_{H/\Gamma} \int_H \int_H K^{\mu}(\bar{u}v)\, K^{\mu}(w\bar{u})\, K^{\mu}(wv)\, \overline{\gamma_{\beta}}(u)\, \nu_{\alpha}(v)\, \nu_{\gamma}(w)\, d\sigma(u)\, d\sigma(v)\, d\sigma(w),$$

and to justify the procedure we need only prove uniform convergence.

The triple integral can be rewritten as

$$\int\limits_{H/\Gamma^\mu}\int\limits_H\int\limits_H K(\bar{u}v)\,K(w\bar{u})\,K(\bar{w}v)\,\overline{L^\mu \nu_\beta}\,L^\mu \nu_\alpha\,L^\mu \nu_\gamma\,d\sigma(u)\,d\sigma(v)\,d\sigma(w)\,.$$

Set

$$T_\gamma(uv) = \int\limits_H K(w\bar{u})\,K(\bar{w}v)\,L^\mu \nu_\gamma(w)\,d\sigma(w)\,, \tag{7.6}$$

and let $H(R)$ denote the set $\{v \mid |v| > R\}$. We have to show that

$$\int\limits_{H/\Gamma^\mu}\int\limits_{H(R)} K(\bar{u}v)\,T_\gamma(uv)\,\overline{L^\mu \nu_\beta}\,L^\mu \nu_\alpha\,d\sigma(u)\,d\sigma(v) \tag{7.7}$$

tends uniformly to 0 as $R \to \infty$.

The kernel K has the familiar property (isometry of the Hilbert transform)

$$\int\limits_H \left| \int\limits_H K(z\bar{\zeta})\,h(z)\,d\sigma(z) \right|^2 d\sigma(\zeta) \leqq \int\limits_H |h(z)|^2\,d\sigma(z)\,.$$

We apply this inequaliiy to (7.6) and observe that $L^\mu \nu$ is bounded for small values of $\|\mu\|_\infty$. In this way we obtain

$$\int\limits_H |T_\gamma(uv)|^2\,d\sigma(v) \leqq C^2 \int\limits_H |K(w\bar{u})|^2\,d\sigma(w)\,, \tag{7.8}$$

where C is a constant, and hence

$$\int\limits_{H(R)} |K(\bar{u}v)\,T_\gamma(uv)|\,d\sigma(u)\,d\sigma(v)$$

$$\leqq C \left(\int\limits_{H(R)} |K(\bar{u}v)|^2\,d\sigma(v) \right)^{1/2} \left(\int\limits_H |K(w\bar{u})|^2\,d\sigma(w) \right)^{1/2}.$$

The first factor on the right tends uniformly to 0, and the second factor remains bounded, when u ranges over a compact set. This proves that (7.7) tends uniformly to 0, and (7.5) is established.

We note at once that (7.5) is symmetric in α and γ. In other words,

$$\frac{\partial g_{\alpha\bar{\beta}}}{\partial t_\gamma} = \frac{\partial g_{\gamma\bar{\beta}}}{\partial t_\alpha}\,, \tag{7.9}$$

which means that the metric is Kählerian. In (A) this property was proved

rather indirectly, namely by showing that there exist geodesic coordinates at each point. Indeed, if the ν_α are of the special form $\overline{\varphi}_\alpha y^2$ it was found, almost miraculously, that all the derivatives $\partial g_{\alpha\overline{\beta}} / \partial t_\gamma$ vanish for $t = 0$. According to (7.5) it must consequently be true that

$$\int_{H/\Gamma}\int_H\int_H \frac{K(\overline{u}v)\,K(w\overline{u})\,K(\overline{w}v)}{K(\overline{u}u)\,K(\overline{v}v)\,K(\overline{w}w)}\,\varphi_\beta(u)\,\overline{\varphi}_\alpha(v)\,\overline{\varphi}_\gamma(w)\,d\sigma(u)\,d\sigma(v)\,d\sigma(w) = 0\,, \tag{7.10}$$

which by use of the representation (4.7) is equivalent to

(7.11)

$$\int_{H/\Gamma}\int_H\int_H \frac{K(\overline{u}v)\,K(\overline{w}u)\,K(\overline{w}v)\,K(\overline{u}a)^2\,K(\overline{v}b)^2\,K(\overline{w}c)^2}{K(\overline{u}u)\,K(\overline{v}v)\,K(\overline{w}w)}\,d\sigma(u)\,d\sigma(v)\,d\sigma(w) = 0\,.$$

Our efforts to verify this formula by direct computation have been unsuccessful. It seems quite likely that the method might lead to still other nontrivial identities of the same nature.

8. It is clear that higher derivatives can be computed in the same manner. For instance, we obtain by (7.4) and (7.5)

$$\frac{\partial^2 g_{\alpha\overline{\beta}}}{\partial t_\gamma\,\partial \overline{t}_\delta} = \frac{24}{\pi^3}\int_{H/\Gamma\times H^3} [K(\overline{z}u)\,K(\overline{z}v)\,K(\overline{w}u)\,K(\overline{w}v)$$

$$+ K(\overline{u}v)\,K(\overline{z}w)\,K(\overline{z}u)\,K(\overline{w}v) + K(\overline{u}v)\,K(\overline{w}u)\,K(\overline{z}w)\,K(\overline{z}v)] \tag{8.1}$$

$$\overline{\nu}_\beta(u)\,\nu_\alpha(v)\,\nu_\gamma(w)\,\overline{\nu}_\delta(z)\,d\sigma(u)\,d\sigma(v)\,d\sigma(w)\,d\sigma(z)\,.$$

In this formula we have omitted the superscripts, thereby indicating that we are mainly interested in the values at $t = 0$. For arbitrary t the derivatives are obtained by replacing each K with the corresponding K^μ.

The convergence is proved exactly as before. For instance, with the notation (7.6) the first term is

$$\int_{H/\Gamma} \overline{\nu}_\beta(u)\,d\sigma(u)\int_H \overline{T}_\delta(vu)\,T_\gamma(uv)\,\nu_\alpha(v)\,d\sigma(v)$$

and we find by use of the estimate (7.8) that the integrand converges uniformly when u is restricted to a fundamental region.

Again, it is irrelevant which variable ranges over H/Γ. For reasons of symmetry we prefer to write (8.1) in the form

$$
\begin{aligned}
\frac{\partial^2 g_{\alpha\bar\beta}}{\partial t_\gamma \partial \bar t_\delta} = \frac{24}{\pi^3} \int\limits_{H/\Gamma\times H^3} & [K(\bar w \bar u) K(\bar w z) K(v \bar u) K(vz) \\
(8.2) \qquad & + K(\bar u z) K(\bar w v) K(\bar w \bar u) K(vz) + K(\bar u z) K(v \bar u) K(\bar w v) K(\bar w z)] \\
& \gamma_\alpha(z) \bar\gamma_\beta(u) \gamma_\gamma(v) \bar\gamma_\delta(w)\, d\sigma(z)\, d\sigma(u)\, d\sigma(v)\, d\sigma(w).
\end{aligned}
$$

In order to draw the conclusions we shall need it is desirable to write (8.2) in still another form which results by use of the reproduction formula

$$
(8.3) \qquad K(u\bar v) = -\frac{1}{\pi} \int\limits_H K(a\bar v) K(\bar a u)\, d\sigma(a).
$$

We apply this to $K(\bar w z)$, $K(v\bar u)$ in the first term, to $K(\bar u z)$, $K(\bar w v)$ in the second term, and to $K(v\bar u)$, $K(\bar w z)$ in the third term on the right hand side of (8.2). In this way we obtain

$$
\begin{aligned}
\frac{\partial^2 g_{\alpha\bar\beta}}{\partial t_\gamma \partial \bar t_\delta} = \frac{24}{\pi^5} \int\limits_{H/\Gamma\times H^5} & [K(\bar w \bar u) K(\bar w a) K(\bar a z) K(v \bar b) K(b \bar u) K(v z) \\
(8.4) \qquad & + K(\bar u a) K(\bar a z) K(\bar w b) K(\bar b v) K(\bar w \bar u) K(vz) \\
& + K(\bar u z) K(v \bar a) K(a \bar u) K(\bar w v) K(\bar w b) K(\bar b z)] \\
& \gamma_\alpha(z) \bar\gamma_\beta(u) \gamma_\gamma(v) \bar\gamma_\delta(w)\, d\sigma(z)\, d\sigma(u)\, d\sigma(v)\, d\sigma(w)\, d\sigma(a)\, d\sigma(b).
\end{aligned}
$$

Once more, the integration variables can be interchanged in an arbitrary manner. We choose to let a range over H/Γ, the others over H. Then, if we introduce the notations

$$
\begin{aligned}
& L_{\alpha\beta}(ab) = \int\limits_{H^2} K(uv) K(u\bar a) K(v\bar b) \gamma_\alpha(u) \gamma_\beta(v)\, d\sigma(u)\, d\sigma(v) \\
(8.5) \qquad & \\
& L_{\alpha\bar\beta}(ab) = \int\limits_{H^2} K(u\bar v) K(u\bar a) K(\bar v b) \gamma_\alpha(u) \bar\gamma_\beta(v)\, d\sigma(u)\, d\sigma(v)
\end{aligned}
$$

we find that

$$
\begin{aligned}
\frac{\partial^2 g_{\alpha\bar\beta}}{\partial t_\gamma \partial \bar t_\delta} = \frac{24}{\pi^5} \int\limits_{H/\Gamma\times H} & [L_{\alpha\gamma}(ab) \bar L_{\delta\beta}(ab) \\
(8\ 6) \qquad & + L_{\alpha\gamma}(ab) \bar L_{\beta\delta} + L_{\alpha\bar\beta}(ab) \bar L_{\delta\bar\gamma}(ab)]\, d\sigma(a)\, d\sigma(b).
\end{aligned}
$$

A more symmetric result is obtained when we observe that

$$L_{\alpha\beta}(ab) = L_{\beta\alpha}(ba),$$

and that we may interchange a and b in any term. The final formula reads

$$\partial^2 g_{\alpha\bar\beta} / \partial t_\gamma \partial \bar t_\delta = \tag{8.7}$$
$$= \frac{12}{\pi^5} \int_{H/\Gamma \times H} [(L_{\alpha\gamma} + L_{\gamma\alpha})(\bar L_{\beta\delta} + \bar L_{\delta\beta}) + 2 L_{\alpha\bar\beta} \bar L_{\delta\bar\gamma}] \, d\sigma(a) \, d\sigma(b),$$

all arguments being ab.

9. We recall some fundamental notions of differential geometry as applied to Kählerian manifolds (see Bochner [6]). By agreement, Greek indices will run from 1 to n, while Latin indices will run through $1, \ldots, n, \bar 1, \ldots, \bar n$. We make use of the summation convention whenever applicable.

The fundamental metric tensor is extended to all pairs of subscripts by setting $g_{\alpha\beta} = g_{\bar\alpha\bar\beta} = 0$ and $g_{\bar\beta\alpha} = g_{\alpha\bar\beta}$. To conform with Bochner's notation we write $ds^2 = 2 g_{\alpha\bar\beta} \, dt_\alpha \, d\bar t^\beta$. The matrix $\| g^{ij} \|$ is the inverse of $\| g_{ij} \|$; hence $g_{\alpha\bar\beta} g^{\gamma\bar\beta} = 0$ if $\alpha \neq \gamma$, 1 if $\alpha = \gamma$. As usual, we set

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{li}}{\partial t_j} + \frac{\partial g_{li}}{\partial t_i} - \frac{\partial g_{ij}}{\partial t_l} \right) \tag{9.1}$$

and we find that these quantities are 0 unless all indices are barred, or all unbarred. For unbarred indices we obtain, in the Kählerian case,

$$\Gamma^\gamma_{\alpha\beta} = g^{\gamma\bar\varepsilon} \frac{\partial g_{\alpha\bar\varepsilon}}{\partial t_\beta} = g^{\gamma\bar\varepsilon} \frac{\partial g_{\beta\bar\varepsilon}}{\partial t_\alpha}. \tag{9.2}$$

From now on we assume, for simplicity, that the coordinates are geodesic and orthonormal at the origin. Then the curvature tensor (in Bochner's notation) is given by

$$R^i_{jkl} = - \frac{\partial \Gamma^i_{jk}}{\partial t_l} + \frac{\partial \Gamma^i_{jl}}{\partial t_k} \tag{9.3}$$

and, except for conjugation, its only nonzero components are

$$R^\alpha_{\beta\gamma\bar\delta} = - R^\alpha_{\beta\bar\delta\gamma} = - \frac{\partial^2 g_{\beta\bar\alpha}}{\partial t_\gamma \partial \bar t_\delta}. \tag{9.4}$$

When the index is lowered we obtain

$$R_{\alpha\bar\beta\gamma\bar\delta} = R^{\bar\alpha}_{\bar\beta\gamma\bar\delta} = \frac{\partial^2 g_{\alpha\bar\beta}}{\partial t_\gamma \, \partial \bar t_\delta}, \tag{9.5}$$

and the contracted curvature tensor, defined by

$$R_{ik} = g^{jl} R_{ijkl}, \tag{9.6}$$

has the components

$$R_{\alpha\bar\beta} = -\frac{\partial^2 g_{\alpha\bar\delta}}{\partial t_\delta \, \partial \bar t_\beta} = -\frac{\partial^2 g_{\alpha\bar\beta}}{\partial t_\delta \, \partial \bar t_\delta}. \tag{9.7}$$

The scalar curvature is

$$R = R_{\alpha\bar\alpha} = -\frac{\partial^2 g_{\alpha\bar\alpha}}{\partial t_\delta \, \partial \bar t_\delta}. \tag{9.8}$$

Let ϑ be a selfadjoint vector of unit length at the origin, i.e., suppose that $\vartheta^{\bar\alpha} = \bar\vartheta^\alpha$ and $\vartheta^\alpha \bar\vartheta^\alpha = 1$. The Ricci curvature in the direction ϑ is then given by

$$R_{\alpha\bar\beta}\, \vartheta^\alpha \bar\vartheta^\beta = -\frac{\partial^2 g_{\alpha\bar\beta}}{\partial t_\delta \, \partial \bar t_\delta}\, \vartheta^\alpha \bar\vartheta^\beta . \tag{9.9}$$

A holomorphic section is spanned by two selfadjoint unit vectors with unbarred components λ^α and $i\lambda^\alpha$. According to Bochner ([6], Theorem 4) the curvature of a holomorphic section is

$$-R_{\alpha\bar\beta\gamma\bar\delta}\, \lambda^\alpha \bar\lambda^\beta \lambda^\gamma \bar\lambda^\delta . \tag{9.10}$$

10. In the case of Weil's metric on T_g it is now easy to prove:

Theorem. *The Ricci curvatures, the curvatures of holomorphic sections, and the scalar curvature are negative.*

Indeed, with the help of (8.7) and (9.9) we find the following expression for the Ricci curvature:

$$R_{\alpha\bar\beta}\, \vartheta^\alpha \bar\vartheta^\beta =$$

$$= -\frac{12}{\pi^5} \int\limits_{H/\Gamma \times H} [(L_{\alpha\delta}+L_{\delta\alpha})(\bar L_{\delta\beta}+\bar L_{\beta\delta}) + 2L_{\alpha\bar\delta}\bar L_{\beta\bar\delta}]\, \vartheta^\alpha \bar\vartheta^\beta \, d\sigma(a)\, d\sigma(b) \tag{10.1}$$

$$= -\frac{12}{\pi^5} \sum_\delta \int\limits_{H/\Gamma \times H} \left(\Big| \sum_\alpha (L_{\alpha\delta}+L_{\delta\alpha})\, \vartheta^\alpha \Big|^2 + 2 \Big| \sum_\alpha L_{\alpha\bar\delta}\, \vartheta^\alpha \Big|^2 \right) d\sigma(a)\, d\sigma(b).$$

Similarly, the sectional curvature is given by

$$(10.2)\quad -\frac{12}{\pi^5}\int\limits_{H/\Gamma\times H}\left(\left|\sum_{\alpha,\gamma}(L_{\alpha\gamma}+L_{\gamma\alpha})\lambda^\alpha\lambda^\gamma\right|^2+2\left|\sum_{\alpha,\beta}L_{\alpha\bar\beta}\lambda^\alpha\bar\lambda^\beta\right|^2\right)d\sigma(a)\,d\sigma(b)\,.$$

Both expressions are clearly ≤ 0.

To see that the curvatures are strictly negative we observe that the vanishing of (10.1) would imply

$$\sum_\alpha L_{\alpha\bar\delta}\,\vartheta^\alpha=0$$

for all δ, identically in a and b. We choose $b=\bar a$ which gives

$$\sum_\alpha\left(\vartheta^\alpha\int\limits_{H^2}K(u\bar v)\,K(ua)\,K(\overline{va})\,\gamma_\alpha(u)\,\overline{\gamma_\delta}(v)\,d\sigma(u)\,d\sigma(v)\right)=0\,.$$

Integrate with respect to a over H/Γ. Because of the reduction formula (8.3) we obtain

$$\sum_\alpha\left(\vartheta^\alpha\int\limits_{H/\Gamma\times H}K(u\bar v)^2\,\gamma_\alpha(u)\,\overline{\gamma_\delta}(v)\,d\sigma(u)\,d\sigma(v)\right)=0\,,$$

and this is nothing else than $g_{\alpha\bar\delta}\,\vartheta^\alpha=0$, an absurd relation. The proof is similar in the case of (10.2).

It is a corollary that the scalar curvature is negative, for $R=R_{\alpha\bar\alpha}$ is a sum of Ricci curvatures. The explicit formula reads

$$(10.3)\quad R=-\frac{12}{\pi^5}\sum_{\alpha,\delta}\int\limits_{H/\Gamma\times H}\left(|L_{\alpha\delta}+L_{\delta\alpha}|^2+2\,|L_{\alpha\bar\delta}|^2\right)d\sigma(a)\,d\sigma(b)\,.$$

REFERENCES

[1] L. Ahlfors, On quasiconformal mappings, *Journal d'Analyse Mathématique*, 3, 1953—4.

[2] L. Ahlfors, The complex analytic structure of the space of closed Riemann surfaces, Analytic functions, Princeton 1960.

[3] L. Ahlfors, Some remarks on Teichmüller's space of Riemann surfaces, *Annals of Math.* 72, 1960.

[4] L. Ahlfors — L. Bers. Riemann's mapping theorem for variable metrics, *Annals of Math.* 74, 1960.

[5] L. Bers, Spaces of Riemann surfaces, *Proc. Intern. Congr. of Math.*, Edinburgh 1958.

[6] S. Bochner, Curvature in Hermitian metric. *Bull. Amer. Math. Soc.*, 53, 1947.

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(Received February 16, 1961)

VARIATION DIMINISHING TRANSFORMATIONS AND ORTHOGONAL POLYNOMIALS [1]

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1. Introduction.

Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomials in the usual normalization, that is if $\alpha > -1$, $\beta > -1$

$$2^n n!\, P_n^{(\alpha,\beta)}(x) = (-1)^n (1-x)^{-\alpha} (1+x)^{-\beta} \left(\frac{d}{dx}\right)^n [(1-x)^{\alpha+n} (1+x)^{\beta+n}].$$

These polynomials satisfy the orthogonality relations

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)\, P_m^{(\alpha,\beta)}(x)\, \Omega_{\alpha,\beta}(x)\, dx = h_{\alpha,\beta}(n)\, \delta_{n,m}$$

where

$$h_{\alpha,\beta}(n) = \frac{2^{\alpha+\beta+1}\, \Gamma(n+\alpha+1)\, \Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\, n!\, \Gamma(n+\alpha+\beta+1)}, \tag{1}$$

and

$$\Omega_{\alpha,\beta}(x) = (1-x)^{\alpha} (1+x)^{\beta}.$$

Let $l^2_{\alpha,\beta}$ consist of those real functions $f(n)$ defined for $n = 0, 1, 2, \ldots$ for which

$$\|f\|_2 = \left[\sum_{n=0} |f(n)|^2\, h_{\alpha,\beta}(n)\right]^{1/2} < \infty, \tag{2}$$

and let $L^2_{\alpha,\beta}$ consist of those real measurable functions $\varphi(x)$ on $-1 < x < 1$ for which

$$\|\varphi\|_2 = \left[\int_{-1}^{1} \varphi(x)^2\, \Omega_{\alpha,\beta}(x)\, dx\right]^{1/2} < \infty.$$

1. Research supported in part by the Air Force Office of Scientific Research and Development Command under contract No. AF 49 (638)—218.

For $f \in l^2_{\alpha,\beta}$ we set

$$f^\frown(x) = \sum_{n=0}^{\infty} f(n) P_n^{(\alpha,\beta)}(x). \tag{3}$$

It follows from the orthogonality relations above that the series (3) converges in the mean in $L^2_{\alpha,\beta}$ and that

$$\| f^\frown(x) \|_2 = \| f(n) \|_2 .$$

Let $M(x)$ be a real bounded measurable function defined on $-1 \leqq x \leqq 1$. If for $f \in l^2_{\alpha,\beta}$ we define

$$T_M f \cdot (n) = [h_{\alpha,\beta}(n)]^{-1} \int_{-1}^{1} f^\frown(x) P_n^{(\alpha,\beta)}(x) M(x) \Omega_{\alpha,\beta}(x)\, dx \tag{4}$$

then it is evident that T_M is a bounded transformation of $l^2_{\alpha,\beta}$ into itself; moreover

$$\| T_M \| = \| M(x) \|_\infty$$

where $\| M(x) \|_\infty$ is the essential least upper bound of $|M(x)|$ on $-1 \leqq x \leqq 1$. Transformations of the form T_M will be called multiplier transformations in $l^2_{\alpha,\beta}$.

Let us denote by $\mathbf{V}[f]$ the number of changes of sign of $f(n)$ for $n = 0, 1, \ldots$. $M(x)$ will be called a variation diminishing multiplier in $l^2_{\alpha,\beta}$ if for every $f \in l^2_{\alpha,\beta}$

$$\mathbf{V}[T_M f] \leqq \mathbf{V}[f].$$

In the present paper we will show that $M(x)$ is variation diminishing in $l^2_{\alpha,\beta}$ if and only if it is of the form

$$M(x) = d e^{cx} \frac{\prod_1^\infty (1 + a_k x)}{\prod_1^\infty (1 - b_k x)}$$

where $c \geqq 0$, $1 \geqq a_k \geqq 0$, $1 > b_k > 0$ and $\Sigma_k (a_k + b_k) < \infty$. Analogous results are also obtained for Laguerre and Hermite polynomials.

In earlier papers the author dealt with the same problems for Hankel transforms and for ultraspherical polynomials. In both these cases there

exists a convolution structure. In the present paper these problems are taken up from a different point of view using methods which do not depend upon the existence of a convolution. The method used here is very general and depends very little on the special properties of the classical orthogonal polynomials. It will, for example, certainly work for any orthogonal set of polynomials relative to a fairly well behaved weight function on a finite interval. The method is also applicable to some Sturm-Liouville systems.

2. The theorems of Schoenberg and Edrei.

In this section we reproduce two basic results which we will need later.

Definition. A matrix[2] $[a(n, k)]$ $n_1<n<n_2$, $k_1<k<k_2$ is said to be variation diminishing if

$$\psi(n) = \sum_{k_1<k<k_2} a(n, k)\, \varphi(k) \qquad n_1<n<n_2$$

implies $\mathbf{V}[\psi] \leqq \mathbf{V}[\varphi]$ for every function $\varphi(k)$ which is zero except for finitely many k.

Theorem 2b. (Schoenberg). If $a(u)$ is a measurable real function defined for $-\infty<u<\infty$ such that for every positive integer n and real numbers $s_1<s_2<\ldots<s_n$, $r_1<r_2<\ldots<r_n$, the matrix $[a(r_i-s_j)]$ is variation diminishing and if $\lim\limits_{u\to\pm\infty} a(u)=0$ then

$$a(u) = \int_{-\infty}^{\infty} \left[\delta e^{\gamma x^2+i\beta x} \prod_k (1+i\,\alpha_k x)\, e^{-i\alpha_k x}\right]^{-1} e^{ixu}\, du$$

where all the constants $\alpha_k, \beta, \gamma, \delta$ are real and in addition $\gamma \geqq 0$ $\sum\limits_k \alpha_k^2 < \infty$.

Theorem 2c. (Edrei). If $a(n)$ is a real function defined for $n=0, \pm 1, \pm 2, \ldots$ such that $[a(n-k)]$ $-\infty<n$, $k<\infty$ is a variation diminishing matrix and if $\lim\limits_{n\to\pm\infty} a(n)=0$ then

2. Here n_1 and k_1 may be finite or $-\infty$ and n_2 and k_2 may be finite or $+\infty$.

$$a(n) = \int_{-\pi}^{\pi} \zeta \left[\exp i(m-n)\vartheta + \varepsilon_1 e^{i\vartheta} + \varepsilon_{-1} e^{-i\vartheta}\right] \frac{\prod_k (1+\alpha_k e^{i\vartheta}) \prod_k (1+\beta_k e^{-i\vartheta})}{\prod_k (1-\gamma_k e^{i\vartheta}) \prod_k (1-\delta_k e^{-i\vartheta})} \, d\vartheta .$$

Here the α's, β's, γ's, δ's, ε's and ζ are real and in addition: $1 \geqq \alpha_k > 0$, $1 \geqq \beta_k > 0$, $1 > \gamma_k > 0$, $1 > \delta_k > 0$,

$$\Sigma_k (\alpha_k + \beta_k + \gamma_k + \delta_k) < \infty, \quad \varepsilon_1 \geqq 0, \quad \varepsilon_{-1} \geqq 0 .$$

Finally m is an integer.

For the demonstrations of these theorems see the papers of Schoenberg and Edrei listed in the bibliography.

3. Jacobi polynomials.

We recall that the Jacobi polynomials satisfy the recursion relation

$$(1) \qquad P_{n+1}^{(\alpha,\beta)}(x) = (A_n x + B_n) P_n^{(\alpha,\beta)}(x) - C_n P_{n-1}^{(\alpha,\beta)}(x) ,$$

where

$$\begin{aligned} &2(n+1)(n+\alpha+\beta+1) A_n = (2n+\alpha+\beta+1)(2n+\alpha+\beta+2) , \\ (2) \quad &2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta) B_n = (\alpha^2-\beta^2)(2n+\alpha+\beta+1) , \\ &(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta) C_n = (n+\alpha)(n+\beta)(2n+\alpha+\beta+2) . \end{aligned}$$

We define the operator Δ by

$$\Delta f(n) = [C_{n+1} / A_{n+1}] f(n+1) + [1 / A_{n-1}] f(n-1) - [B_n / A_n] f(n).$$

Here $f(-1)$ is to be taken as 0. Let I be the identity operator

$$If \cdot (n) = f(n) .$$

Lemma 3a. *If $\xi > 1$ then*

$$\mathbf{V}[(\Delta - \xi I) f] \geq \mathbf{V}[f]$$

for every $f \in l^2_{\alpha,\beta}$.

To prove this let us introduce the operators

$$\begin{aligned} \delta_+ f \cdot (n) &= f(n+1) - f(n) \qquad & n = 0, 1, \ldots ; \\ \delta_- f \cdot (n) &= f(n) - f(n-1) \qquad & n = 0, 1, \ldots ; \end{aligned}$$

here $f(-1)$ is to be taken as 0. We further define the functions

$$\alpha(n) = [P_n^{(\alpha,\beta)}(\xi)]^{-1} ,$$

$$\beta(n) = P_{n+1}^{(\alpha,\beta)}(\xi)\, P_n^{(\alpha,\beta)}(\xi) \left[\prod_{k=0}^{n} C_k\right]^{-1},$$

$$\gamma(n) = \left[\prod_{k=0}^{n} C_k\right] A_n^{-1}\, P_n^{(\alpha,\beta)}(\xi)^{-1}.$$

We assert that

$$(3) \qquad \Delta - \xi I = (\alpha I)\,\delta_-\,(\beta I)\,\delta_+\,(\gamma I).$$

It is evident that

$$(\alpha I)\,\delta_-\,(\beta I)\,\delta_+\,(\gamma I)\,\varphi = \varphi(n+1)\,[\alpha(n)\,\beta(n)\,\gamma(n+1)]$$
$$- \varphi(n)\,[\alpha(n)\,\beta(n)\,\gamma(n) + \alpha(n)\,\beta(n-1)\,\gamma(n)]$$
$$+ \varphi(n-1)\,[\alpha(n)\,\beta(n-1)\,\gamma(n-1)],$$

so that proving (3) amounts to verifying that for $n = 0, 1, \ldots$

$$\alpha(n)\,\beta(n)\,\gamma(n+1) = C_{n+1}\,A_{n+1}^{-1},$$
$$\alpha(n)\,\beta(n-1)\,\gamma(n-1) = A_{n-1}^{-1},$$
$$\alpha(n)\,\beta(n)\,\gamma(n) + \alpha(n)\,\beta(n-1)\,\gamma(n) = \xi + B_n A_n^{-1}.$$

With the help of the recursion formula these identities can easily be checked. It is evident that if $\zeta(n) > 0$ $n = 0, 1, \ldots$ then for every real function $\varphi(n)$ defined for $n = 0, 1, \ldots$

$$\mathbf{V}\,[\delta_-\,(\zeta I)\,\varphi] \geqq \mathbf{V}\,[\varphi].$$

If $\eta(n) > 0$ and if in addition $\eta(n)\,\varphi(n) \to 0$ as $n \to \infty$ then

$$\mathbf{V}\,[\delta_+\,(\eta I)\,\varphi] \geqq \mathbf{V}\,[\varphi].$$

Since $\| f(n) \|_2$ is finite it follows that $| f(n) |^2\, h_{\alpha,\beta}(n) \to 0$ as $n \to \infty$ and thus that $f(n) = O(\sqrt{n})$ as $n \to \infty$. Now

$$P_n^{(\alpha,\beta)} = \left(\frac{x+1}{2}\right)^n \sum_{m=0}^{n} \binom{n+\alpha}{n-m} \binom{n+\beta}{m} \left(\frac{x-1}{x+1}\right)^m \geqq \left(\frac{x+1}{2}\right)^n \binom{n+\alpha}{n} \qquad \text{if } x \geqq 1,$$

see [4; vol. 2, 10.8 (12)]. Since

$$\prod_{0}^{n} C_k = \frac{(2n+\alpha+\beta+2)\,\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+2)\,(n+1)\,!} \cdot S$$

where S is a constant independent of n, we have

$$\prod_{0}^{n} C_k = O(n^{-1})$$

and since $A_n^{-1} = O(1)$ it follows that

$$\gamma(n) = O[n^{-\alpha-1} 2^n (\xi + 1)^{-n}] \qquad n \to \infty .$$

Thus $\gamma(n) f(n) \to 0$ as $n \to \infty$. The various remarks we have made taken together prove our lemma.

Theorem 3b. *The multiplier* $M(x) = (x-\xi)^{-1}$ *where* $\xi > 1$, *is variation diminishing in* $l^2_{\alpha,\beta}$.

It is evident from (1) that

$$(\Delta - \xi I) T_M f \cdot (n) = f(n).$$

By Lemma 2a

$$\begin{aligned} \mathbf{V}[f] &= \mathbf{V}[(\Delta - \xi I) T_M f] \\ &\geqq \mathbf{V}[T_M f] \end{aligned}$$

as desired.

Lemma 3c. *If* $\xi > 1$ *then for any real function* $f(n)$ $n = 0, 1, \ldots,$

$$V[(\Delta + \xi I) f] \leqq V[f].$$

We introduce the operators

$$\left.\begin{aligned} \sigma_+ f \cdot (n) &= f(n+1) + f(n) \\ \sigma_- f \cdot (n) &= f(n) + f(n-1) \end{aligned}\right. \qquad n = 0, 1, \ldots;$$

here again $f(-1)$ is to be taken as zero. We assert that

$$(\Delta + \xi I) = (\alpha I) \sigma_- (\beta I) \sigma_+ (\gamma I).$$

This can be verified exactly as before. If $\zeta(n) > 0$ then for every φ

$$\begin{aligned} \mathbf{V}[\sigma_+ (\zeta I) \varphi] &\leqq \mathbf{V}[\varphi] \\ \mathbf{V}[\sigma_- (\zeta I) \varphi] &\leqq \mathbf{V}[\varphi]. \end{aligned}$$

See Pólya and Szegö [8; vol. 2, p. 38]. Our lemma is an immediate consequence of these assertions.

Theorem 3d. *The multiplier* $M(x) = (x + \xi)$, *where* $\xi > 1$, *is variation diminishing in* $l^2_{\alpha,\beta}$.

It is evident that in this case

$$T_M f \cdot (n) = (\Delta + \xi I) f \cdot (n)$$

etc.

Lemma 3e. If $M_1(x)$ and $M_2(x)$ are variation diminishing multipliers on $l^2_{\alpha,\beta}$ then so is $M(x) = M_1(x) M_2(x)$.

This is because $T_M f = T_{M_1} [T_{M_2} f]$. (The verification of this relation depends on the fact that the $P_n^{(\alpha,\beta)}$ are complete in $L^2{}_{\alpha,\beta}$.)

Theorem 3f. Let $c > 0$,

$$1 \geqq a_1 \geqq a_2 \geqq \dots \geqq 0, \quad \text{and} \quad 1 > b_1 \geqq b_2 \geqq \dots \geqq 0$$

where $\Sigma_k (a_k + b_k) < \infty$. If

$$M(x) = e^{cx} \frac{\prod_k (1 + a_k x)}{\prod_k (1 - b_k x)}$$

then $M(x)$ is a variation diminishing multiplier on $l^2_{\alpha,\beta}$.

If $M(x)$ is of the form

$$M(x) = \frac{\prod_{k=1}^{N} (1 + a_k x)}{\prod_{k=1}^{M} (1 - b_k x)}$$

then $M(x)$ is variation diminishing by Theorems 2b and 2d and Lemma 3e. In the general case let

$$M_r(x) = \frac{\prod_{k=1}^{r} (1 + a_k x)}{\prod_{k=1}^{r} (1 - b_k x)} \cdot \left[1 + \frac{x}{cr}\right]^r.$$

It is evident that $M_r(x) \to M(x)$ uniformly on $-1 \leqq x \leqq 1$ as $r \to \infty$. This implies, using Parseval's theorem, that $\| T_M f - T_{M_r} f \|_2 \to 0$ as $r \to \infty$ and thus that $T_{M_r} f \cdot (n) \to T_M f \cdot (n)$ for each n. It follows that

$$\mathrm{V}[T_M f] \leqq \varliminf_{r \to \infty} \mathrm{V}[T_{M_r} f] \leqq \varliminf_{r \to \infty} \mathrm{V}[f] = \mathrm{V}[f].$$

We now turn to the problem of establishing the converse result.

Theorem 3g. If $M(x)$ is a variation diminishing multiplier in $l^2_{\alpha,\beta}$ and if

$$m(k) = \frac{1}{\pi} \int_0^\pi M(\cos\vartheta) \cos k\vartheta \, d\vartheta \qquad -\infty < k < \infty$$

then

$$[m(k-j)] \qquad -\infty < k, j < \infty$$

is a variation diminishing matrix.

If $f(n) \in l^2_{\alpha,\beta}$ is zero except for a finite number of integers n then

$$T_M f \cdot (n) = \sum_{m=0}^{\infty} f(m) h_{\alpha,\beta}(n)^{-1} \int_{-1}^{1} M(x) P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x) \Omega_{\alpha,\beta}(x) \, dx$$

$$= \sum_{m=0}^{\infty} h_{\alpha,\beta}(n)^{-1/2} K(n, m) h_{\alpha,\beta}(m)^{1/2} f(m)$$

where $K(n, m)$ is

$$[h_{\alpha,\beta}(n) h_{\alpha,\beta}(m)]^{-1/2} \int_0^\pi M(\cos\vartheta) P_n^{(\alpha,\beta)}(\cos\vartheta) P_m^{(\alpha,\beta)}(\cos\vartheta)$$

$$\cdot\, \Omega_{\alpha,\beta}(\cos\vartheta) \sin\vartheta \, d\vartheta .$$

The following asymptotic formula is demonstrated in Szegö [12; § 8.21].

(4) $\quad P_n^{(\alpha,\beta)}(\cos\vartheta) =$

$$= \sqrt{2/\pi}\, h_{\alpha,\beta}(n)^{1/2} [\Omega_{\alpha,\beta}(\cos\vartheta) \sin\vartheta]^{-1/2} \cos\{N(n)\vartheta + \gamma\} + O(n^{-3/2})$$

where $N(n) = n + (\alpha + \beta + 1)/2$, $\gamma = -(\alpha + 1/2)\pi$. The bound for the error in (4) holds uniformly in the interval $\varepsilon < \vartheta < \pi - \varepsilon$ for any fixed $\varepsilon > 0$. Formally we have

(5) $\quad \lim_{k\to\infty} K(k+r, k+s)$

$$= \lim_{k\to\infty} \frac{2}{\pi} \int_0^\pi M(\cos\vartheta) \cos\{N(k+r)\vartheta + \gamma\} \cos\{N(k+s)\vartheta + \gamma\} \, d\vartheta$$

$$= \lim_{k\to\infty} \frac{1}{\pi} \int_0^\pi M(\cos\vartheta) [\cos(r-s)\vartheta + \cos\{[N(k+r) + N(k+s)]\vartheta + 2\gamma\}] \, d\vartheta$$

$$= \frac{1}{\pi} \int_0^\pi M(\cos\vartheta) \cos(r-s)\vartheta \, d\vartheta ,$$

by the Riemann-Lebesgue lemma. This argument fails to be rigorous in that (4) does not hold uniformly in the entire interval $0 \leq \vartheta \leq \pi$. In what follows we shall show by a simple argument that the endpoints 0 and π do not matter. We have if $\varepsilon > 0$

$$\lim_{n \to \infty} h_{\alpha,\beta}(n)^{-1} \int_{\varepsilon}^{\pi-\varepsilon} [P_n^{(\alpha,\beta)}(\cos\vartheta)]^2 \, \Omega_{\alpha,\beta}(\cos\vartheta) \sin\vartheta \, d\vartheta$$

$$= \lim_{n \to \infty} \frac{1}{\pi} \int_{\varepsilon}^{\pi-\varepsilon} [\cos\{N(n)\vartheta + \gamma\}]^2 \, d\vartheta = (\pi - 2\varepsilon)/\pi .$$

Since

$$h_{\alpha,\beta}(n)^{-1} \int_0^{\pi} [P_n^{(\alpha,\beta)}(\cos\vartheta)]^2 \, \Omega_{\alpha,\beta}(\cos\vartheta) \sin\vartheta \, d\vartheta = 1$$

for $n = 0, 1, \ldots,$ it follows that

$$\lim_{n \to \infty} h_{\alpha,\beta}(n)^{-1} \left\{ \int_0^{\varepsilon} + \int_{\pi-\varepsilon}^{\pi} \right\} [P_n^{(\alpha,\beta)}(\cos\vartheta)]^2 \, \Omega_{\alpha,\beta}(\cos\vartheta) \sin\vartheta \, d\vartheta = 2\varepsilon/\pi .$$

An evident application of Schwarz's inequality now yields

$$\overline{\lim_{k \to \infty}} \left| K(k+r, k+s) - \frac{1}{\pi} \int_{\varepsilon}^{\pi-\varepsilon} M(\cos\vartheta) \cos\{(r-s)\vartheta\} \, d\vartheta \right| \leq \|M\|_\infty \frac{2\varepsilon}{\pi} .$$

Letting $\varepsilon \to 0+$ we obtain (5), this time rigorously. Since obviously "variation diminishing matrices go into variation diminishing matrices in the limit" our theorem follows.

Theorem 3h. *If $M(x)$ is a variation diminishing multiplier in $l^2_{\alpha,\beta}$ then*

$$M(x) = d e^{cx} \frac{\prod_k (1 + a_k x)}{\prod_k (1 - b_k x)}$$

where d is real, $c \geqq 0$, $1 \geqq a_k > 0$, $1 > b_k > 0$, and $\Sigma_k (a_k + b_k) < \infty$.

By Theorem 3g the matrix $[m(k-j)]$ is variation diminishing, and by Riemann-Lebesgue lemma $m(k) \to 0$ as $k \to \pm\infty$. By Edrei's theorem

we have, taking into account the fact that $m(k)$ is even,

$$(6) \qquad m(k) = \int_{-\pi}^{\pi} \left\{ \zeta \exp\left[\varepsilon e^{i\vartheta} + \varepsilon e^{-i\vartheta}\right] \frac{\prod_{1}^{\infty}(1-\alpha_k e^{i\vartheta})(1-\alpha_k e^{-i\vartheta})}{\prod_{1}^{\infty}(1+\beta_k e^{i\vartheta})(1+\beta_k e^{-i\vartheta})} \right\} e^{-ik\vartheta}\, d\vartheta$$

where ζ is real, $\varepsilon \geqq 0$, $1 \geqq \alpha_k > 0$, $1 > \beta_k > 0$, $\Sigma_k (\alpha_k + \beta_k) < \infty$. Setting $2\varepsilon = c$, $2\alpha_k / (1 + \alpha_k^2) = a_k$, $2\beta_k / (1 + \beta_k)^2 = b_k$, etc., this can be rewritten in the form

$$m(k) = \frac{1}{\pi} \int_0^{\pi} d \exp\left[c \cos\vartheta\right] \frac{\prod_k (1 - a_k \cos\vartheta)}{\prod_k (1 + b_k \cos\vartheta)} \cos k\vartheta \, d\vartheta .$$

Recalling the definition of $m(k)$ we see that our desired result is a consequence of this relation.

4. Laguerre polynomials.

In this section we shall establish results for generalized Laguerre polynomials analogous to those previously established for Jacobi polynomials. Interestingly enough both the results and the details of the proof are significantly different. For $\alpha > -1$ let

$$L_n^{(\alpha)}(x) = (n!)^{-1} e^x x^{-\alpha} (d/dx)^n \left[e^{-x} x^{n+\alpha}\right] \qquad n = 0, 1, \ldots .$$

We have the orthogonality relations

$$\int_0^{\infty} x^{\alpha} e^{-x} L_n^{(\alpha)}(x) L_m^{(\alpha)} dx = \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + 1)} \delta_{n,m} .$$

We denote by l_α^2 the set of those real functions $f(n)$ defined for $n = 0, 1, \ldots$ for which

$$\|f\|_2 = \left[\sum_0^{\infty} f(n)^2 \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + 1)}\right]^{1/2}$$

is finite. If we denote by L_α^2 the set of those real functions $\varphi(x)$ for which

$$\|\varphi\|_2 = \left[\int_0^{\infty} \varphi(x)^2 e^{-x} x^{\alpha} dx\right]^{1/2}$$

is finite, and if for $f \in l^2_\alpha$ we set

$$f^\frown(x) = \sum_{n=0}^{\infty} f(n)\, L_n^{(\alpha)}(x)$$

then $f^\frown \in L^2_\alpha$ and $\|f\|_2 = \|f^\frown\|_2$. As before let $M(x)$ be a bounded measurable function defined for $0 \leq x < \infty$. For $f \in l^2_\alpha$ we set

$$T_M f \cdot (n) = \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \int_0^\infty f^\frown(x)\, M(x)\, L_n^{(\alpha)}(x)\, x^\alpha e^{-x}\, dx .$$

Then T_M is a bounded transformation of l^2_α into itself.

Theorem 4a. *If $M(x)$ is of the form*

$$M(x) = d\left[e^{cx} \prod_1^\infty (1 + a_k x)\right]^{-1} \tag{1}$$

where d is real, $c \geq 0$, $a_k \geq 0$ $k = 0, 1, \ldots,$ and $\sum_1^\infty a_k < \infty$, then $M(x)$ is variation diminishing.

It is enough to prove that $(\xi + x)^{-1}$, $\xi > 0$, is a variation diminishing multiplier, since the result above can then be obtained from this special case by a now routine argument.

The $L_n^{(\alpha)}(x)$ satisfy the recurrence formula

$$(n+1)\, L_{n+1}^{(\alpha)}(x) = (-x + 2n + 1 + \alpha)\, L_n^{(\alpha)}(x) - (n+\alpha)\, L_{n-1}^{(\alpha)}(x) .$$

We define the operator Δ

$$\Delta f \cdot (n) = (n + 1 + \alpha)\, f(n+1) + n f(n-1) - (2n + 1 + \alpha)\, f(n) .$$

Let

$$\varphi(n) = [L_n^{(\alpha)}(-\xi)]^{-1}$$
$$\chi(n) = L_n^{(\alpha)}(-\xi)\, L_{n+1}^{(\alpha)}(-\xi)\, \{(1)_{n+1} / (\alpha)_{n+1}\}$$
$$\phi(n) = [L_n^{(\alpha)}(-\xi)]^{-1}\, \{(\alpha)_{n+1} / (1)_n\} .$$

We assert that if $\xi > 0$ then

$$(\Delta - \xi I)\, f \cdot (n) = [\varphi I\, \delta_-\, \chi I\, \delta_+ \phi I]\, f \cdot (n) .$$

We omit the verification of this identity. From this it is easily proved that $(\xi + x)^{-1}$ is a variation decreasing multiplier.

Theorem 4b. *If $M(x)$ is a variation decreasing multiplier on l^2_α then $M(x)$ is of the form* (1).

It is evident that the product of variation diminishing multipliers is again variation diminishing. Thus $M_1(x) = e^{-x} M(x)$ is variation diminishing if $M(x)$ is. We set

$$K(n, m) = \int_0^\infty M_1(x) L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) x^\alpha e^{-x} dx .$$

Note that if $f(n) \neq 0$ for only finitely many values of n then

$$T_M f \cdot (n) = \sum_{m=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} K(n, m) f(m) .$$

It is clear that $[K(n, m)]$ $0 \leq n, m < \infty$ is a variation diminishing matrix. Let N be a positive integer and r and s any integers. We define

$$m_N(r, s) = \lim_{k \to \infty} (Nk)^{1-2\alpha} K(N^2 k^2 + kr, N^2 k^2 + ks) . \tag{2}$$

We will prove that this limit exists and that

$$m_N(r, s) = \frac{1}{\pi} \int_0^\infty M_1(x) \cos\left\{\left(\frac{r-s}{N}\right) x^{1/2}\right\} x^{-1/2} \cdot dx . \tag{3}$$

Let $\omega > 0$ be given. We set

$$(Nk)^{1-2\alpha} K(N^2 k^2 + kr, N^2 k^2 + ks) = I_1 + I_2 + I_3$$

where

$$I_1 = (Nk)^{1-2\alpha} \int_0^\omega M_1(x) L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) x^\alpha e^{-x} dx ,$$

$$I_2 = (Nk)^{1-2\alpha} \int_\omega^{k^2} M_1(x) L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) x^\alpha e^{-x} dx ,$$

$$I_3 = (Nk)^{1-2\alpha} \int_{k_2}^\infty M_1(x) L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) x^\alpha e^{-x} dx .$$

Here $n = N^2 k^2 + kr$, $m = N^2 k^2 + ks$.

By Theorem 8.22.1 of [12]

$$\begin{aligned} L_q^{(\alpha)}(x) = {} & \pi^{-1/2} e^{x/2} x^{-\alpha/2-1/4} q^{\alpha/2-1/4} \cos\{2(qx)^{1/2} - \alpha\pi/2 - \pi/4\} \\ & + o(1) \qquad \text{as} \quad q \to \infty \end{aligned} \tag{4}$$

for $0 < x < \infty$. By Theorems 7.6.5 and 8.22.8 of [12] we see that for $0 \leq x \leq q$

$$(5) \qquad | L_q^{(\alpha)}(x) | \leq B e^{x/2} x^{-\alpha/2-1/4} q^{\alpha/2-1/4}$$

where B is a constant independent of x and q. Using Lebesgue's convergence theorem and the Riemann-Lebesgue theorem we find that

$$(6) \qquad \lim_{k\to\infty} I_1 = \frac{1}{2\pi} \int_0^\infty M_1(x) \cos\left\{\left(\frac{r-s}{N}\right) x^{1/2}\right\} x^{-1/2}\, dx\,.$$

Next

$$(7) \qquad \overline{\lim_{k\to\infty}} \, | I_2 | \leq \| M \|_\infty B^2 \int_0^\infty e^{-x} x^{-1/2}\, dx\,.$$

Finally using Schwarz's inequality we find that

$$| I_3 | \leq (Nk)^{1-2\alpha} e^{-k^2} \| M \|_\infty \| L_n^{(\alpha)} \|_2 \| L_m^{(\alpha)} \|_2$$

from which it follows that

$$(8) \qquad \lim_{k\to\infty} I_3 = 0\,.$$

The relations (6), (7) and (8) taken in conjunction prove (3). Let us set

$$(9) \qquad m(u) = \frac{2}{\pi} \int_0^\infty M_1(x^2) \cos ux\, dx\,.$$

We have proved that for every integer n the matrix

$$\left[m\left(\frac{r-s}{N}\right)\right] \qquad -\infty < r, s < \infty$$

is variation diminishing. Since m is continuous it follows that if

$$r_1 < r_2 < \dots < r_p$$
$$s_1 < s_2 < \dots < s_p$$

then the matrix

$$[m(r_i - s_j)] \qquad i, j = 1, \dots, p$$

is variation diminishing.

Applying Theorem 2b to $m(u)$ and taking into account the fact that

$m(u)$ is even we find that

$$(10) \qquad m(u) = \frac{2}{\pi} \int_0^\infty d\left[e^{c_1 x^2} \prod_k (1 + a_k x^2)\right]^{-1} \cos xu \, du$$

where $c_1 = \gamma$, $a_k = \alpha_k^2$, etc. Comparing (9) and (10) we find that

$$M_1(x^2) = d\left[e^{c_1 x^2} \prod_k (1 + a_k x^2)\right]^{-1},$$

$$M(x) \;\; = d\left[e^{cx} \;\; \prod_k (1 + a_k x) \;\right]^{-1}$$

where $c = c_1 - 1$. Since M is by assumption bounded it is evident that $c \geqq 0$.

5. Hermite polynomials.

Let

$$H_n(x) = (-1)^n e^{x^2} (d / dx)^n e^{-x^2} \qquad n = 0, 1, \dots .$$

Then

$$\int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-x^2} dx = \pi^{1/2} 2^n n! \, \delta_{n,m}.$$

If we denote by l^2 the set of those real functions $f(n)$ for which

$$\|f\|_2 = \left[\sum_{n=0}^{\infty} f(n)^2 \pi^{1/2} 2^n n!\right]^{1/2} < \infty,$$

and if we denote by L^2 the set of these measurable real functions $\varphi(x)$ for which

$$\|\varphi\|_2 = \left[\int_{-\infty}^{\infty} \varphi(x)^2 e^{-x^2} dx\right]^{1/2} < \infty$$

then for $f \in l^2$ the series

$$f^\frown(x) = \sum_{n=0}^{\infty} f(n) H_n(x)$$

converges in L^2 to a function $f^\frown(x)$ in L_2 and $\|f\|_2 = \|f^\frown\|_2$. Let $M(x)$ be a bounded measurable function for $-\infty < x < \infty$; we define T_M in the

usual way, etc. The following result asserts that here the only variation diminishing multipliers are the trivial ones, if we restrict slightly the class of multipliers considered.

Lemma 5a. Let $M(x)$ be a real bounded measurable function for $-\infty < x < \infty$ and let in addition:

1. $\int_{-\infty}^{\infty} |M(x)|\, dx < \infty$.

2. $M(x) = o(|x|^{-1/2}) \qquad x \to \pm\infty$.

Then if $M(x)$ is variation diminishing in l^2 it must be identically zero.

If we set

$$K(m, n) = \int_{-\infty}^{\infty} M(x) H_n(x) H_m(x) e^{-x^2}\, dx$$

then $[K(m, n)]$ $0 \leq m, n < \infty$ is a variation diminishing matrix. Let

$$n = N^2 k^2 + kr, \quad m = N^2 k^2 + ks.$$

Then we assert that if

$$\lambda_n = \Gamma(n+1) / \Gamma\left(\frac{n}{2} + 1\right)$$

we have

$$\lim_{\substack{k \to \infty \\ k \in C_i}} \lambda_n^{-1} \lambda_m^{-1} K(m, n) = I_i \quad i = 1, 2, 3, 4 \tag{1}$$

where C_i is the set of integers congruent to i modulo 4 and

$$I_i = 1/2 \int_{-\infty}^{\infty} M(x) \cos\left\{\left(\frac{r-s}{\sqrt{2N}}\right) x - (r-s)\frac{i\pi}{2}\right\} dx. \tag{2}$$

To establish (1) and (2) we recall that

$$\lambda_n^{-1} e^{-x^2/2} H_n(x) - \cos\left\{\sqrt{2n+1}\, x - \frac{n\pi}{2}\right\} = o(1) \tag{3}$$

as $n \to \infty$ for each x, $-\infty < x < \infty$; see [12; § 8.22]. On the other hand the relations

$$H_{2m} = (-1)^m\, 2^{2m}\, m!\, L_m^{(-1/2)}(x^2),$$
$$H_{2m+1}(x) = (-1)^m\, 2^{m+1}\, m!\, x\, L_m^{(1/2)}(x^2),$$

see [12; § 5.6], together with (5) of § 4 imply that if $0 < x < 1/2\sqrt{n}$

$$\lambda_n^{-1} \left| e^{-x^2/2} H_n(x) \right| \leq B, \tag{4}$$

where B is independent of n and x. With (3) and (4) available the demonstration of (1) and (2) can be completed along the lines of § 4.

Let

$$m_e(u) = \int_{-\infty}^{\infty} M(x) \cos ux\, dx.$$

Let $r_1 < r_2$ and $s_1 < s_2$. Let ε_1 and ε_2 each be 0 or 1. Choose integers $r_k^{(N)}$, $s_j^{(N)}$ $k, j = 1, 2$ such that

$$r_k^{(N)} \equiv \varepsilon_k \pmod 2 \qquad s_j^{(N)} \equiv 0 \pmod 2,$$
$$r_k^{(N)}/\sqrt{2}\,N \to r_k \qquad s_j^{(N)}/\sqrt{2}\,N \to s_j \quad \text{as} \quad N \to \infty.$$

Then using (1) and (2) with $i = 2$ we see that the matrix

$$\begin{bmatrix} (-1)^{\varepsilon_1} m_e(r_1 - s_1) & (-1)^{\varepsilon_1} m_e(r_1 - s_2) \\ (-1)^{\varepsilon_2} m_e(r_2 - s_1) & (-1)^{\varepsilon_2} m_e(r_2 - s_2) \end{bmatrix}$$

must be variation diminishing no matter how ε_1 and ε_2 are chosen. This is possible only if $m_e(u) \equiv 0$. A similar argument shows that if

$$m_0(u) = \int_{-\infty}^{\infty} M(x) \sin ux\, dx$$

then $m_0(u) \equiv 0$. These together imply $M(u) \equiv 0$ as desired.

Theorem 5b. *Let $M(x)$ be a real bounded measurable function for $-\infty < x < \infty$ and let in addition*

$$M(x) = O(|x|^{-\varepsilon}) \qquad x \to \pm\infty$$

for some $\varepsilon > 0$. Then if $M(x)$ is variation diminishing in l^2 it must be identically zero.

Choose a positive integer n so large that $n\varepsilon > 1$ and apply Lemma 5a to $M(x)^n$.

It seems probable that the only variation diminishing multipliers

$M(x)$ in l^2 are of the form $M(x) \equiv d$. However Theorem 5b falls just short of proving this.

Bibliography

[1] M. Aissen, I. J. Schoenberg and A. Whitney, On the generating function of totally positive sequences I, *Jour. d'Analyse Math.*, 2 (1952), pp. 93—103.

[2] A. Edrei, On the generating functions of totally positive sequences, II, *Jour. d'Analyse Math.*, 2 (1952), pp. 104—109.

[3] A. Edrei, On the generating function of a doubly infinite totally positive sequence, *Trans. Amer. Math. Soc.*, 74 (1953), 367—383.

[4] A. Erdélyi, et al., Higher Transcendental Functions, New York, 1953.

[5] I. I. Hirschman, Jr. Harmonic analysis and ultraspherical polynomials, Symposium on Harmonic Analysis and Related Integral Transforms, Cornell Univ. (1956).

[6] I. I. Hirschman, Jr., Variation diminishing Hankel transforms and Variation diminishing transformations and ultraspherical polynomials, *Jour. d'Analyse Math.*, 8 (1960—1961), pp. 307—336 and 337—360.

[7] I. I. Hirschman, Jr,, and D. V. Widder, The Convolution Transform, Princeton, 1955.

[8] G. Pólya and G. Szegö, Aufgaben und Lehrsätze aus der Analysis, Berlin, 1925.

[9] I. J. Schoenberg, On Pólya frequency functions I, *Jour. d'Analyse Math.*, 1 (1951), pp. 331—374.

[10] I. J. Schoenberg, On Pólya frequency functions, II, *Acta Scientiarum Mathe.-maticarum Szeged*, 12 (1950).

[11] I. J. Schoenberg, Some analytical aspects of the problem of smoothing, Courant Anniversary Volume, New York, 1948, pp. 351—370.

[12] G. Szegö, Orthogonal Polynomials, New York, 1939.

[13] A. Whitney, A reduction theorem for totally positive matrices, *Jour. d'Analyse Math.*, 2 (1952), pp. 88—92.

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(Received April 3, 1961)

This Paper was submitted to the Canadian Journal of Mathematics on July 8, 1960 and was accepted for publication. The author wishes to thank the editors of the Canadian Journal for their kindness in releasing this paper so that it could be published in the present journal, together with two earlier papers on the same subject.

ASYMPTOTIC DEVELOPMENTS III: AGAIN ON THE FUNDAMENTAL THEOREM OF ASYMPTOTIC SERIES [1]

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The fundamental theorem on asymptotic series [2], [3] can be formulated as follows.

Theorem 1: A series $a_0 + a_1 + \ldots$ converges asymptotically if and only if a_h tends asymptotically to zero as h approaches infinity.

In this theorem, the terms $a_h = a_h(\omega)$, where $h = 0, 1, \ldots$, are defined for each point ω of an unbounded set Ω lying in the complex plane or on a Riemann surface. A number or function which is independent of ω is called "fixed". That a_h tends for $h \to \infty$ asymptotically to zero means that there exists a fixed integer $h_0 \geqq 0$ with the property that for each fixed integer $h \geqq h_0$ it is possible to find three positive fixed numbers c_h, γ_h and q_h such that $q_h \to \infty$ as $h \to \infty$ and that each point ω of Ω with $|\omega| \geqq \gamma_h$ satisfies the inequality

$$|a_h| \leqq c_h |\omega|^{-q_h}.$$

Without loss of generality we may assume that q_h is for $h \geqq h_0$ a monotonically nondecreasing function of h, for otherwise we can replace q_h by $q'_h = \min\limits_{l \geqq h} q_l$, where q'_h is a monotonically nondecreasing function $\leqq q_h$ of h which tends to infinity as $h \to \infty$.

That the series $a_0 + a_1 + \ldots$ converges asymptotically means that it is possible to define on Ω a function $s = s(\omega)$ such that

$$s - (a_0 + a_1 + \ldots + a_{h-1})$$

1. This report has been facilitated by a grant (G 5228) from the National Science Foundation, Washington, D.C.

2. J. G. van der Corput, "Asymptotic Developments I (Fundamental Theorems of Asymptotics)", Journal d'Analyse Mathématique, Vol. 4, 1954—56, pp. 341—418.

3. J. G. van der Corput, "Asymptotic Developments II (Generalization of the Fundamental Theorem on Asymptotic Series)", Journal d'Analyse Mathématique, Vol. 5, 1956—57, pp. 315—320.

tends asymptotically to zero as h approaches infinity. A function s with this property is called the asymptotic sum of the series and we write

$$s \sim \sum_{h=0}^{\infty}{}' a_h .$$

Let us consider the special case that Ω is a set formed by real numbers which has a lower bound α but does not possess an upper bound. A function $f(\omega)$ defined on Ω is called differentiable on Ω if for each point ω of Ω and for each sequence of points $\omega^* \neq \omega$ of Ω which tend to ω the fraction $(f(\omega^*) - f(\omega))/(\omega^* - \omega)$ tends to a finite limit ; this limit, called the derivative, is denoted by $f'(\omega)$. If ω is an isolated point of Ω, then each function defined on Ω is differentiable at ω and we may assign any value to the derivative at that point, since it is impossible to find on Ω points $\omega^* \neq \omega$ which tend to ω.

Now we assume that each term $a_h = a_h(\omega)$ $(h = 0, 1, ...)$ is on Ω infinitely often differentiable with respect to ω and that for each fixed integer $k \geqq 0$ the k^{th} derivative $a_h^{(k)}(\omega)$ tends asymptotically to zero as $h \to \infty$. According to the fundamental theorem, it is possible to define on Ω functions $s_k(\omega)$ $(k = 0, 1, ...)$ such that for each fixed integer $k \geqq 0$

$$s_k(\omega) \sim \sum_{h=0}^{\infty}{}' a_h^{(k)}(\omega) .$$

The deeper question arises whether it is possible to define on Ω an infinitely often differentiable function $s(\omega)$ such that for each fixed integer $k \geqq 0$

$$s^{(k)}(\omega) \sim \sum_{h=0}^{\infty}{}' a_h^{(k)}(\omega) . \tag{1}$$

That the answer is affirmative follows from:

Theorem 2: *If the terms $a_h(\omega)$ $(h = 0, 1, ...)$ are infinitely often differentiable on Ω and if for each fixed integer $k \geqq 0$ the k^{th} derivative $a_h^{(k)}(\omega)$ tends for $h \to \infty$ asymptotically to zero, then it is possible to define on Ω an infinitely often differentiable function $s(\omega)$ which satisfies on Ω the order relation (1) for each fixed integer $k \geqq 0$.*

This theorem reminds the reader perhaps of the fact that under certain general conditions an asymptotically convergent series with only analytic terms has an asymptotic sum which is also analytic, but the two said results belong to different branches of mathematics.

In the proof of Theorem 2, I use a fixed monotonically nonincreasing infinitely often differentiable function $\sigma(t)$ $(-\infty < t < \infty)$ which is equal to 1 for $t \leqq 0$ and equal to 0 for $t \geqq 1$. Such a function is called a smoothing function.

For each fixed integer $k \geqq 0$ it is possible to find a fixed number Γ_k with

$$(2) \qquad |\sigma^{(k)}(t)| \leqq \Gamma_k \qquad (k = 0, 1, \ldots).$$

Lemma 1: Let α be a real fixed number. If a function $\varphi(\omega)$ defined for $\omega \geqq \alpha$ is $\geqq 0$ and tends for $\omega \to \infty$ to infinity, then it is possible to find for $\omega \geqq \alpha$ a monotonically nondecreasing function $\psi(\omega) \leqq \varphi(\omega)$ which is $\geqq 0$, infinitely often differentiable, tends for $\omega \to \infty$ to infinity and satisfies the inequalities

$$(3) \qquad |\psi^{(k)}(\omega)| \leqq \Gamma_k \qquad (k = 1, 2, \ldots),$$

where the constants Γ_k denote the numbers occurring in (2).

Proof: Let $\chi(\omega)$ $(\omega \geqq \alpha)$ denote the lower bound of $\varphi(u)$ for $u \geqq \omega$. Then $\chi(\omega)$ is for $\omega \geqq \alpha$ a monotonically nondecreasing function $\geqq 0$ and $\leqq \varphi(\omega)$ which tends for $\omega \to \infty$ to infinity. Let β be an integer $\geqq \alpha + 1$. Put $\psi(h) = 0$ for each integer $h \leqq \beta$; furthermore, for each integer $h > \beta$

$$(4) \qquad \psi(h+1) = \min(\chi(h), \psi(h) + 1).$$

The inequality

$$(5) \qquad \psi(h+1) \geqq \psi(h)$$

holds for each integer h. This is obvious for $h \leqq \beta$. If $h > \beta$ and if $\psi(h) \geqq \psi(h-1)$, then (4) gives

$$\psi(h+1) \geqq \min\{\chi(h-1), \psi(h-1) + 1\} = \psi(h),$$

so that (5) holds for each integer.

The function $\psi(h)$ tends to infinity as the integer h approaches infinity. Indeed for each positive integer p there exists a positive integer $h^* > \beta$ such that $\chi(h) \geqq p$ for each integer $h \geqq h^*$, consequently

$$\psi(h+1) \geqq \min(p\ ,\ \psi(h)+1),$$

hence

$$\psi(h+p) \geqq \min(p\ ,\ \psi(h)+p) = p$$

so that the monotonic function $\psi(h)$ tends to infinity as $h \to \infty$.

For each integer h we put in the interval $h < \omega < h+1$

$$\psi(\omega) = \psi(h) + \{\psi(h+1) - \psi(h)\}\,\sigma(h+1-\omega) \tag{6}$$

where $\sigma(t)$ denotes the smoothing function introduced above. This formula holds also for $\omega = h$ and for $\omega = h+1$, since $\sigma(1) = 0$ and $\sigma(1) = 1$. The function $\psi(\omega)$ $(-\infty < \omega < \infty)$ is a monotonically nondecreasing function of ω with

$$\psi(\omega) \leqq \psi(h+1) \leqq \chi(h) \leqq \chi(\omega) \leqq \varphi(\omega)$$

by (4). Moreover, $\psi(\omega)$ is infinitely often differentiable with

$$\psi^{(k)}(\omega) = (-)^k\{\psi(h+1) - \psi(h)\}\,\sigma^{(k)}(h+1-\omega) \qquad (k \geqq 1)$$

which is in absolute value

$$\leqq \{\psi(h+1) - \psi(h)\}\,\Gamma_k \leqq \Gamma_k$$

according to (2) and (4). This completes the proof.

Lemma 2: *If $\sigma(t)$ and $\psi(\omega)$ are infinitely often differentiable functions with (2) and (3), respectively, then we have for $k = 0, 1, \ldots$ and for each real constant h*

$$\left|\frac{d^k\,\sigma\{h+1-\psi(\omega)\}}{d\omega^k}\right| \leqq C_k \tag{7}$$

where C_k denotes a suitably chosen number depending only on $\Gamma_0, \Gamma_1, \ldots, \Gamma_k$.

Proof: Formula (7) holds for $k = 0$, if we choose $C_0 = \Gamma_0$. Furthermore,

$$\frac{d\sigma\{h+1-\psi(\omega)\}}{d\omega} = -\sigma'\{h+1-\psi(\omega)\}\,\psi'(\omega)$$

is according to (2) and (3) in absolute value $\leqq \Gamma_1^2$, so that (7) holds for $k = 1$, if we choose $C_1 = \Gamma_1^2$. Moreover,

$$\frac{d^2\sigma\{h+1-\psi(\omega)\}}{d\omega^2} = -\sigma''\{h+1-\psi(\omega)\}\{\psi'(\omega)\}^2 - \sigma'\{h+1-\psi(\omega)\}\,\psi''(\omega)$$

is according to (2) and (3) in absolute value $\leqq \Gamma_2 \Gamma_1^2 + \Gamma_1 \Gamma_2$. so that (7) holds for $k = 2$, if we choose $C_2 = \Gamma_1^2 \Gamma_2 + \Gamma_1 \Gamma_2$. Continuing in this way, we obtain the required result.

Proof of Theorem 2.

From the fact that for each fixed integer $k \geqq 0$ the function $a_h^{(k)}(\omega)$ defined on Ω tends for $h \to \infty$ asymptotically to zero, it follows that for each fixed integer $k \geqq 0$ it is possible to find an integer $h_k \geqq 0$ such that the inequality

$$\text{(8)} \qquad \left| a_h^{(k)}(\omega) \right| \leq c_{hk}\, \omega^{-q_{hk}}$$

holds for each fixed integer $h \geqq h_k$ and for each number $\omega \geqq \gamma_{hk}$ belonging to Ω; here c_{hk}, γ_{hk} and q_{hk} denote suitably chosen fixed positive numbers depending on h and k with the property that for each given integer k the exponent q_{hk} tends to infinity as h approaches infinity.

Let C_k $(k = 0, 1, \ldots)$ denote the fixed numbers mentioned in Lemma 2. Clearly it is possible to find a function $\varphi(\omega) \geqq 0$ which tends for $\omega \to \infty$ to infinity such that for each $\omega \geqq \alpha$ (α is the lower bound of Ω) and each integer k with $0 \leqq k < \varphi(\omega)$ the inequalities

$$\text{(9)} \qquad \sum_{0 \leqq h < \varphi(\omega)} \sum_{l=0}^{k} \binom{k}{l} C_{k-l}\, c_{hl} \leqq \omega$$

and

$$\text{(10)} \qquad \max_{0 \leqq h < \varphi(\omega)} \gamma_{hk} \leqq \omega$$

hold. Notice that this condition is certainly satisfied by the numbers ω with $\varphi(\omega) = 0$, because then there does not exist an integer k with $0 \leqq k < \varphi(\omega)$.

According to Lemma 1, we can construct for $\omega \geqq \alpha$ a monotonically nondecreasing function $\psi(\omega) \leqq \varphi(\omega)$ which is $\geqq 0$, infinitely often differentiable, tends for $\omega \to \infty$ to infinity and satisfies the inequalities (3).

After these preliminary remarks, it is easy to show that the function

$$\text{(11)} \qquad s(\omega) = \sum_{h=0}^{\infty} \sigma\{h + 1 - \psi(\omega)\}\, a_h(\omega)$$

defined on Ω, possesses the required properties. Notice that the series occurring in (11) is finite, since $\sigma\{h + 1 - \psi(\omega)\} = 0$ for $h \geqq \psi(\omega)$.

We must prove that $s(\omega)$ is in Ω infinitely often differentiable and that for each fixed integer $k \geqq 0$

$$s^{(k)}(\omega) - \sum_{h=0}^{H-1} a_h^{(k)}(\omega) \tag{12}$$

tends asymptotically to zero as $H \to \infty$.

Let H and k denote fixed integers $\geqq 0$ with

$$H \geqq 1 + \max_{0 \leqq l \leqq k} h_l .$$

Choose ω in Ω so large that

$$\omega \geqq 1; \quad \psi(\omega) > H+1 \quad \text{and} \quad \psi(\omega) > k .$$

Since $\sigma\{h+1-\psi(\omega)\} = 1$ for $h \leqq H-1 < \psi(\omega)-1$, expression (12) is according to (11) equal to

$$\sum_{H \leqq h < \psi(\omega)} \sigma\{h+1-\psi(\omega)\}\, a_h^{(k)}(\omega) + \sum_{l=0}^{k-1} \binom{k}{l} \sum \left(\frac{d^{k-l}\sigma\{h+1-\psi(\omega)\}}{d\omega^{k-l}} \right) a_h^{(l)}(\omega),$$

where the last sum contains at most one term $\neq 0$, namely the term with $h = \eta$, where η satisfies $\psi(\omega) - 1 \leqq \eta < \psi(\omega)$. It follows from (10) that $\gamma_{hl} \leqq \omega$ for each choice of the integers h and l with $0 \leqq h < \psi(\omega)$ and $0 \leqq l \leqq k < \psi(\omega) \leqq \varphi(\omega)$.

In each term of the sum $\sum\limits_{H \leqq h < \psi(\omega)}$ we have $h \geqq H \geqq h_k$, so that (8) holds. In each term of the sum $\sum\limits_{l=0}^{k-1}$ we have

$$h = \eta \geqq \psi(\omega) - 1 \geqq H \geqq h_l$$

so that according to (8)

$$\left| a_\eta^{(l)}(\omega) \right| \leqq c_{\eta l}\, \omega^{-q_{\eta l}} \qquad (1 \leqq l \leqq k-1).$$

Consequently, (12) is by (7) in absolute value

$$\leqq C_0 \sum_{H \leqq h < \psi(\omega)} c_{hk}\, \omega^{-q_{hk}} + \sum_{l=0}^{k-1} \binom{k}{l} C_{k-l}\, c_{\eta l}\, \omega^{-q_{\eta l}} .$$

Since q_{hl} is a monotonic function of h, it follows from $H \leqq h$ and $H \leqq \eta$ that $q_{hk} \geqq q_{Hk}$ and $q_{\eta l} \geqq q_{Hl}$, so that (12) is in absolute value

$$\leq \omega^{-m_{Hk}} \left\{ C_0 \sum_{H \leq h < \psi(\omega)} c_{hk} + \sum_{l=0}^{k-1} \binom{k}{l} C_{k-l} c_{\eta l} \right\}$$

where

$$m_{Hk} = \min_{0 \leq l \leq k} q_{Hl} . \tag{13}$$

The expression between braces is in virtue of (9) and $\psi(\omega) \leq \varphi(\omega)$ at most equal to 2ω, so that (12) is in absolute value $\leq 2\omega^{1-m_{Hk}}$. From definition (13), it follows that for a given integer $k \geq 0$ the exponent m_{Hk} tends to infinity as $H \to \infty$, so that (12) tends for $H \to \infty$ asymptotically to zero. This completes the proof.

Theorem 2 contains as special case:

Theorem 3: *If $a_0, a_1, \ldots$ are arbitrary fixed complex numbers, then there exists a function $f(x)$ infinitely often differentiable for $x \geq 0$ with*

$$f^{(k)}(0) = a_k \qquad (k = 0, 1, \ldots) .$$

Proof: We apply the preceding theorem in the special case that Ω consists of the positive numbers ω. In the application, we replace the functions $a_h(\omega)$ $(h = 0, 1, \ldots)$ of Theorem 2 by the expressions

$$\frac{a_h}{h!} \omega^{-h} \qquad (h = 0, 1, \ldots)$$

where the numbers a_h $(h = 0, 1, \ldots)$ are those given in the hypothesis above. In this way we find for $\omega > 0$ an infinitely often differentiable function $s(\omega)$ with

$$s^{(l)}(\omega) \sim \sum_{h=0}^{\infty}{}' \frac{a_h}{h!} \frac{d^l \omega^{-h}}{d\omega^l} \qquad (l = 0, 1, \ldots).$$

We shall show that the function $f(x)$ defined by $f(0) = a_0$ and $f(x) = s(\frac{1}{x})$ $(x > 0)$ possesses the required properties. Put $\omega = \frac{1}{x}$ for $x > 0$. We have for each choice of the fixed integers $k \geq 0$ and $l \geq 0$

$$\frac{d^l f^{(k)}(x)}{d\omega^l} \sim \sum_{h=0}^{\infty}{}' \frac{a_{h+k}}{h!} \frac{d^l \omega^{-h}}{d\omega^l} . \tag{14}$$

This is obvious for $k = 0$. If $k \geq 1$ and if (14) holds with k replaced by $k - 1$, then

$$f^{(k)}(x) = \frac{d}{dx} f^{(k-1)}(x) = -\omega^2 \frac{d}{d\omega} f^{(k-1)}(x)$$

so that the left side of (14) is equal to

$$-\omega^2 \frac{d^{l+1} f^{(k-1)}(x)}{d\omega^{l+1}} - 2l\omega \frac{d^l f^{(k-1)}(x)}{d\omega^l} - l(l-1) \frac{d^{l-1} f^{(k-1)}(x)}{d\omega^{l-1}}$$

$$\sim -\sum_{h=0}^{\infty}{}' \frac{a_{h+k-1}}{h!} \left\{ \omega^2 \frac{d^{l+1}\omega^{-h}}{d\omega^{l+1}} + 2l\omega \frac{d^l \omega^{-h}}{d\omega^l} + l(l-1) \frac{d^{l-1}\omega^{-h}}{d\omega^{l-1}} \right\}$$

$$\sim -\sum_{h=0}^{\infty}{}' \frac{a_{h+k-1}}{h!} \frac{d^l}{d\omega^l} \left(\omega^2 \frac{d\omega^{-h}}{d\omega} \right)$$

$$\sim \sum_{h=0}^{\infty}{}' \frac{a_{h+k-1}}{h!} h \frac{d^l \omega^{1-h}}{d\omega^l} \sim \sum_{h=0}^{\infty}{}' \frac{a_{h+k}}{h!} \frac{d^l \omega^{-h}}{d\omega^l}.$$

This yields (14), in particular for $l = 0$

$$f^{(k)}(x) \sim \sum_{h=0}^{\infty}{}' \frac{a_{h+k}}{h!} x^h,$$

so that $f^{(k)}(x) \to a_k$ as $x \to 0$ and

$$\frac{f^{(k)}(x) - a_k}{x} \sim \sum_{h=1}^{\infty}{}' \frac{a_{h+k}}{h!} x^{h-1} \to a_{k+1}$$

as $x \to 0$, hence $f^{(k+1)}(0) = a_{k+1}$. This completes the proof.

Remark: If $f(x)$ and $g(x)$ are infinitely often differentiable in an interval $\alpha \leqq x \leqq \beta$ with

$$f^{(k)}(\alpha) = a_k \quad \text{and} \quad g^{(k)}(\beta) = b_k \qquad (k = 0, 1, \ldots)$$

and if $\sigma(t)$ is the smoothing function introduced above, then

$$\varphi(x) = \sigma\left(\frac{x-\alpha}{\beta-\alpha}\right) f(x) + \sigma\left(\frac{\beta-x}{\beta-\alpha}\right) g(x)$$

is in the interval $\alpha \leqq x \leqq \beta$ infinitely often differentiable with

$$\varphi^{(k)}(\alpha) = a_k \quad \text{and} \quad \varphi^{(k)}(\beta) = b_k \qquad (k = 0, 1, \ldots).$$

The function $s(\omega)$ occurring in Theorem 2 is not uniquely defined. On the contrary, if β is an arbitrary fixed number $> \alpha$ and if $g(\omega)$ defined at the points ω of Ω with $\alpha \leqq \omega \leqq \beta$ is infinitely often differentiable, then it is possible to construct a function $s(\omega)$ with the required properties

which coincides with $g(\omega)$ at the points ω of Ω with $\alpha \leqq \omega \leqq \beta$. Indeed, if $\gamma > \beta$ is a number belonging to Ω, then according to the preceding remark, it is possible to construct an infinitely often differentiable function $\psi(\omega)$ $(\beta \leqq \omega \leqq \gamma)$ with

$$\psi^{(k)}(\beta) = g^{(k)}(\beta) \quad \text{and} \quad \psi^{(k)}(\gamma) = s^{(k)}(\gamma) \qquad (k = 0, 1, \ldots)$$

so that the function defined on Ω which is equal to $g(\omega)$ in the interval $\alpha \leqq \omega \leqq \beta$, equal to $\psi(\omega)$ in the interval $\beta \leqq \omega \leqq \gamma$ and equal to $s(\omega)$ for $\omega \geqq \gamma$, possesses the required properties.

This leads to the following question: If we know that a function $s(\omega)$ is infinitely often differentiable for $\omega \geqq 0$, and if the asymptotic behavior of each derivative $s^{(k)}(\omega)$, where k denotes a fixed integer $\geqq 0$, is known, what do we know then about the behavior of the function $s(\omega)$ in the interval $0 \leqq \omega \leqq 10^{10}$? The answer is: "nothing", apart from the fact that the function is infinitely often differentiable in that interval.

The following theorem is obvious:

Theorem 4: *Let r be a fixed integer $\geqq 0$. If the terms $a_h(\omega)$ $(h = 0, 1, \ldots)$ defined on Ω are r times differentiable with respect to ω and if for $k = 0, 1, \ldots, r$ the function $a_h^{(k)}(\omega)$ tends asymptotically to zero as h approaches infinity, then it is possible to construct on Ω an r-times differentiable function $s(\omega)$ which satisfies (1) for $k = 0, 1, \ldots, r$.*

This theorem remains true if the two expressions "differentiable" are replaced by "continuously differentiable".

The proof is the same as that of Theorem 2, apart from the simplification that in the proof of Theorem 4 we restrict ourselves to the integers $k \geqq 0$ which are $\leqq r$.

Until now we have restricted ourselves to one variable ω, but the argument given above can also be applied on functions of two or more real independent variables $\omega_1, \ldots, \omega_m$, provided that the notions of asymptotic equality and of asymptotic limits are defined in a natural way. Let Ω be a region in a real m-dimensional space such that each point $(\omega_1, \ldots, \omega_m)$ belonging to Ω satisfies the inequalities

$$\omega_\mu \geqq \alpha_\mu \qquad (\mu = 1, \ldots, m)$$

where $\alpha_1, \ldots, \alpha_m$ denote given fixed numbers. We assume moreover that Ω

contains at least one sequence of points $(\omega_1, \ldots, \omega_m)$ such that each of the m coordinates $\omega_1, \ldots, \omega_m$ tends to infinity.

Finally we introduce a set K in which each element is a system $(k_1, \ldots, k_m)$ formed by m integers $\geqq 0$ with the following property: if K contains a system $(k'_1, \ldots, k'_m)$, then it contains each system $(k_1, \ldots, k_m)$ formed by m integers with $0 \leqq k_\mu \leqq k'_\mu$ $(\mu = 1, \ldots, m)$.

Theorem 5: Assume that the functions $a_h(\omega_1, \ldots, \omega_m)$ $(h = 0, 1, \ldots)$ defined at each point $(\omega_1, \ldots, \omega_m)$ of the region Ω have the property that for each fixed system $(k_1, \ldots, k_m)$ belonging to K the function

$$\frac{\partial^{k_1 + \ldots + k_m} a_h(\omega_1, \ldots, \omega_m)}{\partial\omega_1^{k_1} \ldots \partial\omega_m^{k_m}}$$

exists, is continuous and tends asymptotically to zero as $h \to \infty$.

Then there exists a function $s(\omega_1, \ldots, \omega_m)$ defined in Ω such that for each fixed system $(k_1, \ldots, k_m)$ belonging to K

$$\frac{\partial^{k_1 + \ldots k_m} s(\omega_1, \ldots, \omega_m)}{\partial\omega_1^{k_1} \ldots \partial\omega_m^{k_m}}$$

exists, is continuous and is asymptotically equal to

$$\sum_{h=0}^{\infty}{}' \frac{\partial^{k_1 + \ldots + k_m} a_h(\omega_1, \ldots, \omega_m)}{\partial\omega_1^{k_1} \ldots \partial\omega_m^{k_m}}.$$

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(Received November 22, 1960)

THE NEUTRALIZED SUM FORMULA OF EULER

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CHAPTER 1. NEUTRICES

1.1. Introduction. This paper facilitated by Grant G5288 of the National Science Foundation (Washington, D.C., U. S. A.) includes the contents of a lecture given on August 17, 1960 at Stanford (California) in a schedule of public lectures on numerical analysis, programming theory, and related mathematics being sponsored jointly by the Applied Mathematics Department, Lockheed Missiles and Space Division and by the Numerical Analysis Research Project of Stanford's Applied Mathematics and Statistics Laboratories.

This paper is self contained and can be used as an introduction to the neutrix calculus, independently of the other papers [1)] written by me on this theory. In these previous papers I have developed an abstract theory of neutrices without being much concerned with its applications, but in the present paper devoted to the sum formula of Euler, I follow the opposite way. Here a neutrix and a neutrix relation are only introduced if they lead to a generalization of the sum formula. A symbolism and a new technique are developed which in difficult, perhaps otherwise unassailable problems yield, with little effort, the asymptotic behavior of a great number of integrals and sums. It is my intention to apply the method in following communications on multiple integrals and multiple sums and also on sums of another type than those treated in this paper.

The results obtained are consequences of the theory of neutrices. The application of the neutrices on the sum formula of Euler is not the only

1) Neutrices, J. Soc. Indust. Appl. Math., 7, No. 3, September 1959, 253—279.

Introduction to the Neutrix calculus, reports 128, 129 and 130 of the Mathematics Research Center at Madison; Journal d'Analyse Mathématique, 7, 1959—60, 281—399.

Neutrix calculus I, Neutrices and distributions, Report 142 of the Mathematics Research Center at Madison; Proceedings Royal Neth. Acad. of Sciences 63, series A, 1960, 115—123; Indagationes Mathematicae 22, 1960, 115—123.

Neutrix calculus II, Special neutrix calculus, Report 143 of the Mathematics Research Center at Madison. This communication will appear in the Proceedings of the Royal Neth. Acad. of Sciences.

Neutrix calculus III, General neutrix calculus, Report 144 of the Mathematics Research Center at Madison.

Distributions with compatible neutrices, Report 166 of the Mathematics Research Center at Madison. Journal d'Analyse Mathématique, 8, 1960/61, 185—207.

Introduction to the residue calculus, technical report Lockheed LMSD—703063, July 1960. This communication will appear in the Proceedings of the Royal Neth. Acad. of Sciences.

purpose of the neutrix calculus. Several branches of analysis can be generalized by means of neutrices in the same way as it has been done in this paper for the sum formula of Euler.

The neutrix calculus is based on the fact that functions of a certain type may be neglected. This idea is not new and occurs in a simple form in the works of several mathematicians, for instance in Hadamard's book [2] on the Cauchy problem. This mathematician defines the finite part of an integral by neglecting powers of $x-a$ and even functions of the form $(x-a)^\sigma \log(x-a)$. In a paper published in 1957 C. E. Duncan [3] applies this idea to determine, by means of the sum formula of Euler, the asymptotic behavior of certain trigonometric sums.

The aim of this article is to develop a method which yields asymptotic expansions for an extended class of sums. Several of the expansions obtained by this method are not only asymptotically convergent but at the same time convergent in the usual sense; this fact, however, is irrelevant for the purpose of this paper.

1.2. The primitive sum formula of Euler. This formula can be formulated as follows.

Theorem 1.2.1: *If $f(x)$ is at least m times $(m \geqq 1)$ continuously differentiable in an interval $a \leqq x \leqq b$, then*

$$\sum_{a<n<b} f(n) - \int_a^b f(x)\,dx \tag{1.2.1}$$

$$= \Lambda_m(b-, f) - \Lambda_m(a+, f) + (-)^{m-1} \int_a^b f^{(m)}(x)\,\varphi_m(x)\,dx$$

where

$$\Lambda_m(x, f) = \sum_{h=0}^{m-1} (-)^{h+1} f^{(h)}(x)\,\varphi_{h+1}(x). \tag{1.2,2}$$

Here $\varphi_1(x)$ is the periodic function of x with period 1

2) J. Hadamard, Lectures on Cauchy's problem in linear partial differential equations, Yale and Oxford University Press, 1923, 316 pages; Le problème de Cauchy et les équations partielles linéaires hyperboliques, Paris, Herman et Cie, 1932.

3) C. E. Duncan, On the asymptotic behavior of trigonometric sums, third communication, Proceedings Royal Neth. Acad. of Sciences 60, 1957, 374—380.

which is equal to $x-1/2$ in the interval $0<x<1$ and equal to zero at $x=0$; $\varphi_{h+1}(x)\,(h\geqq 1)$ is the integral of $\varphi_h(x)$ uniquely defined by the condition

$$\int_0^1 \varphi_{h+1}(x)\,dx = 0\,. \tag{1.2.3}$$

The functions $\varphi_{h+1}(x)$ $(h=0, 1, \ldots)$ are periodic functions of x with period 1. This follows for $h=0$ from the definition; if $h\geqq 1$ and $\varphi_h(x)$ has the period 1, then

$$\varphi_{h+1}(x+1)-\varphi_{h+1}(x)=\int_x^{x+1} \varphi_h(t)\,dt = \int_0^1 \varphi_h(t)\,dt = 0 \tag{1.2.4}$$

so that also $\varphi_{h+1}(x)$ has the period 1. The periodic functions $h!\,\varphi_h(x)$ are called the Bernoulli functions.

In this paper the designations $\Lambda_m(x, f)$ and $\varphi_h(x)$ denote always the functions introduced in this section.

The functions $\varphi_2(x), \varphi_3(x), \ldots$ are continuous, but $\varphi_1(x)$ is discontinuous at the numbers which coincide with an integer. If x is not an integer, then

$$\Lambda_m(x\,,\,f) = \Lambda_m(x-\,,\,f) = \Lambda_m(x+\,,\,f)\,, \tag{1.2.5}$$

but if x is an integer, then

$$\Lambda_m(x\,,\,f) = \Lambda_m(x-\,,\,f) + \tfrac{1}{2} f(x) = \Lambda_m(x+\,,\,f) - \tfrac{1}{2} f(x). \tag{1.2.6}$$

The proof of the sum formula is very simple. Repeated integration by parts yields

$$\int_a^b f(x)\,dx = \int_a^b f(x)\,\varphi'_1(x)\,dx = \sum_{a<n<b} f(n) - \int_a^b f'(x)\,\varphi_1(x)\,dx$$
$$+ f(b)\,\varphi_1(b-) - f(a)\,\varphi_1(a+)$$
$$= \sum_{a<n<b} f(n) - \Lambda_m(b-\,,f) + \Lambda_m(a+\,,f) - (-)^{m-1}\int_a^b f^{(m)}(x)\,\varphi_m(x)\,dx$$

and this is the primitive sum formula.

Example: Determine for large positive ω the asymptotic behavior of the sum

$$\sum_{0 < n < \omega} (n+\vartheta)^{-s}$$

where the complex number s and the positive number $\vartheta \leq 1$ are independent of ω and n.

Applying the primitive sum formula of Euler with $-\vartheta < a < 0$; $b = \omega$ and $f(x) = (x+\vartheta)^{-s}$ we obtain

$$\sum_{0 \leq n < \omega} (n+\vartheta)^{-s} - \int_a^\omega (x+\vartheta)^{-s}\, dx$$

$$= \Lambda_m(\omega-, f) - \Lambda_m(a, f) - s(s+1)\dots(s+m-1)\int_a^\omega (x+\vartheta)^{-s-m}\varphi_m(x)\, dx.$$

If we choose $m > -\operatorname{Re} s + 1$, then the last integrand is integrable to infinity, so that for $s \neq 1$

$$\sum_{0 \leq n < \omega} (n+\vartheta)^{-s} - \frac{(\omega+\vartheta)^{1-s}}{1-s} \tag{1.2.7}$$

$$= \Lambda_m(\omega-, f) + s(s+1)\dots(s+m-1)\int_\omega^\infty (x+\vartheta)^{-s-m}\varphi_m(x)\, dx + \chi^{(s)}$$

where

$$\chi^{(s)} = -\frac{(a+\vartheta)^{1-s}}{1-s} - \Lambda_m(a, f) - s(s+1)\dots(s+m-1)\int_a^\infty (x+\vartheta)^{-s-m}\varphi_m(x)\, dx.$$

Consequently $\chi^{(s)}$ represents in the half plane $\operatorname{Re} s > 1-m$, the point $s = 1$ excepted, an analytic function of s. Formula (1.2.7), applied with $\omega \to \infty$, shows that for each $s \neq 1$ the function $\chi^{(s)}$ is the analytic continuation of the function represented in the half-plane $\operatorname{Re} s > 1$ by

$$\sum_{n=0}^{\infty} (n+\vartheta)^{-s} = \zeta(s, \vartheta)$$

so that $\chi^{(s)} = \zeta(s, \vartheta)$ for each $s \neq 1$. Using this in (1.2.7) and letting m tend to infinity we obtain for each fixed $s \neq 1$ the asymptotic expansion

$$\sum_{0 \leq n < \omega} (n+\vartheta)^{-s} \sim \frac{(\omega+\vartheta)^{1-s}}{1-s} + \zeta(s, \vartheta) + \Lambda_\infty(\omega-, f) \tag{1.2.8}$$

where the last term denotes the asymptotically convergent series

$$-\sum_{h=0}^{\infty}{}' s(s+1)\ldots(s+h-1)\,\varphi_{h+1}(\omega-)\,(\omega+\vartheta)^{-s-h}.$$

The prime is used to distinguish the asymptotically convergent series from the convergent series; in Section 2.2 I give the general definition of an asymptotically convergent series.

To obtain the corresponding result for the case $s=1$, we use the fact that (1.2.8) holds uniformly for the points $s \neq 1$ in the neighborhood of 1. We have

$$\frac{(\omega+\vartheta)^{1-s}}{1-s} = \frac{1}{1-s} + \log(\omega+\vartheta) + o(1)$$

where $o(1)$ denotes a function of s which is analytic and equal to 0 at $s=1$. Furthermore

$$\zeta(s,\vartheta) = \frac{1}{s-1} + \Gamma_{\vartheta} + o(1) \tag{1.2.9}$$

where Γ_{ϑ} is a constant. For instance $\zeta(s,1)$ is the zeta function of Riemann and Γ_1 is the constant of Euler. Substitution in (1.2.8) shows that this formula remains valid at $s=1$, provided that we replace, on the right-hand side,

$$\frac{(\omega+\vartheta)^{1-s}}{1-s} + \zeta(s,\vartheta) \quad \text{by} \quad \log(\omega+\vartheta) + \Gamma_{\vartheta}.$$

Notice that for each $s \neq 1$, the asymptotic expansion contains only powers of $\omega+\vartheta$, but that in the case $s=1$ also a logarithmic term occurs.

We shall find in Section 3.6 more general results of the same kind.

1.3. Definition of neutrices. Let N' be an arbitrary non-empty set. A function with domain N' is a function $f(\xi)$ defined for each element ξ of N'. A neutrix N with domain N' is an additive group formed by real or complex functions $f(\xi)$ with domain N' which satisfies the

Neutrix condition: *If the additive group contains a function $\gamma(\xi)$ which is constant (this means: is independent of ξ), then this constant is equal to zero.*

The functions $\nu(\xi)$ belonging to the neutrix are said to be negligible or, more precisely, to be negligible in N.

That the neutrix is an additive group means that any two functions negligible in N have the property that also their sum and their difference are negligible in N.

In my previous papers on neutrices I have defined them in an abstract way, but in the theory of the sum formula of Euler we have only to deal with real or complex functions, so that here we can restrict ourselves to neutrices which are formed by real or complex functions.

If a function $g(\xi)$ with domain N' is, apart from a term negligible in N, equal to a constant, then this constant is uniquely defined. Indeed, if

$$g(\xi) = \gamma + \nu(\xi) = \gamma_1 + \nu_1(\xi)$$

where γ and γ_1 are constants and where $\nu(\xi)$ and $\nu_1(\xi)$ are negligible in the neutrix N, then the negligible function $\nu(\xi) - \nu_1(\xi)$ is equal to the constant $\gamma_1 - \gamma$, so that this constant vanishes according to the neutrix condition imposed on the neutrix N. This uniquely determined constant γ is called the neutralized value which $g(\xi)$ assumes at N and is denoted by $g(N)$. This designation denotes therefore a number independent of ξ which is obtained by canceling in $g(\xi)$ the negligible terms. In this case we say that the neutrix N neutralizes the function $g(\xi)$.

Example: All the functions which can be written in the form

$$p(\xi) + o(1)$$

where $p(\xi)$ denotes a polynomial in ξ without constant term and where $o(1)$ denotes a function of ξ defined for $\xi > 1$ which tends to zero for $\xi \to \infty$, form a neutrix N with domain $1 < \xi < \infty$. Indeed if

$$p(\xi) + o(1) = \gamma \qquad (1 < \xi < \infty),$$

where γ is independent of ξ, then all the coefficients occurring in $p(\xi)$ and also the constant γ are equal to zero, so that N satisfies the neutrix condition. This neutrix has the property that

$$N^5 \sqrt[3]{N^3 + 1} = -\frac{1}{9}$$

since we have for large positive ξ

$$\xi^5\sqrt[3]{\xi^3+1} = \xi^6(1+\xi^{-3})^{1/3} = \xi^6 + \frac{1}{3}\xi^3 - \frac{1}{9} + o(1)$$

where ξ^6, $\frac{1}{3}\xi^3$, and $o(1)$ are negligible in N.

Furthermore

$$\int_1^N x\,dx = -\frac{1}{2}$$

since

$$\int_1^\xi x\,dx = \frac{1}{2}\xi^2 - \frac{1}{2}$$

where $\frac{1}{2}\xi^2$ is negligible in N. On the other hand

$$\int_1^N \frac{dx}{x} \tag{1.3.1}$$

has no meaning, since

$$\int_1^\xi \frac{dx}{x} = \log\xi \tag{1.3.2}$$

can not be written as a constant plus a function which is negligible in N. This means that the neutrix N is too meager for the treatment of the integral (1.3.1). In such a case we enlarge our neutrix; in other words we construct a neutrix E such that each function negligible in N is certainly negligible in E and that (1.3.1) obtains a meaning if we replace N by E. For instance the neutrix E formed by the functions of the form

$$p(\xi) + c\log\xi + o(1)$$

where the coefficients c are arbitrary complex constants, possesses the required property. That E satisfies the neutrix condition, is clear, for if

$$p(\xi) + c\log\xi + o(1) = \gamma \qquad (1 < \xi < \infty)$$

where γ is independent of ξ, then all the coefficients in $p(\xi)$ and also the coefficients c and γ are equal to zero. This neutrix E has the property

$$\int_1^E \frac{dx}{x} = 0$$

since the function $\log \xi$ occurring in (1.3.2) is negligible in E.

In this section I shall examine how in general such an enlargement can be found.

Definition: Consider infinitely many neutrices N_h $(h=0, 1, \ldots)$ with the same domain N' and with the property that each function negligible in N_h $(h=0, 1, \ldots)$ is negligible in N_{h+1}. The union U of these neutrices $N_0, N_1, \ldots$ is formed by the functions which are negligible in at least one of these neutrices. This union is a neutrix with domain N', for if a function $\nu(\xi)$ belonging to U is equal to a constant, then $\gamma(\xi)$ is negligible in at least one neutrix N_h, so that $\gamma=0$ according to the neutrix condition imposed on this neutrix N_h.

Theorem 1.3.1: If $N_0, N_1, \ldots$ are neutrices with the same domain N', and with the property that each function negligible in N_h $(h=0, 1, \ldots)$ is negligible in N_{h+1} and if the function $f_h(\xi)$ $(h=0, 1, \ldots)$ defined on N' assumes at N_h a neutralized value, then $f_h(\xi)$ $(h=0, 1, \ldots)$ assumes at U the same neutralized value.

Proof: We have for each element ξ of N'

$$f_h(\xi)=\gamma_h+\nu_h(\xi)$$

where $\gamma_h=f_h(N_h)$ and where $\nu_h(\xi)$ is negligible in N_h, therefore certainly negligible in U, so that $f_h(\xi)$ assumes at U the neutralized value γ_h.

Theorem 1.3.2: Let N be a neutrix with domain N' such that each function $\nu(\xi)$ negligible in N and each rational number ρ have the property that the product $\rho\nu(\xi)$ is negligible in N. For each function $f(\xi)$ defined on N' it is possible to find a neutrix M with the following three properties:

(1) $f(\xi)$ assumes at M a neutralized value,

(2) each function negligible in N is negligible in M,

(3) each function $\mu(\xi)$ negligible in M and each rational number ρ have the property that the product $\rho\mu(\xi)$ is negligible in M.

Proof: If $f(\xi)$ assumes at N a neutralized value, then $M=N$ has the required property. If $f(\xi)$ does not assume at N a neutralized value, then

I choose for M the set formed by the functions of the form

$$\mu(\xi) = \rho(f(\xi) - \gamma_0) + \nu(\xi) \tag{1.3.3}$$

where γ_0 is a given constant, where the coefficients ρ are arbitrary rational constants and where the terms $\nu(\xi)$ represent arbitrary functions negligible in N. The neutrix condition is satistied. Indeed, if $\mu(\xi) = \gamma$, where γ is constant, then $\rho = 0$, since otherwise

$$f(\xi) = -\frac{\nu(\xi)}{\rho} + \gamma_0 + \frac{\gamma}{\rho}$$

would assume at N the neutralized value $\gamma_0 + \dfrac{\gamma}{\rho}$, contrary to the hypothesis; $\rho = 0$ implies $\nu(\xi) = \gamma$, hence $\gamma = 0$.

By choosing in (1.3.3), $\rho = 1$ and $\nu(\xi) = 0$, we see that $f(\xi) - \gamma_0$ is negligible in M, so that $f(\xi)$ assumes at M the neutralized value γ_0. By choosing in (1.3.3), $\rho = 0$, we see that each function negligible in N is also negligible in M. Finally it follows from (1.3.3) that each function $\mu(\xi)$ negligible in M multiplied with an arbitrary rational number yields a function which is negligible in M. This completes the proof.

Theorem 1.3.3: *Let N be a neutrix with domain N' such that each function $\nu(\xi)$ negligible in N and each rational number ρ have the property that the product $\rho\nu(\xi)$ is negligible in N.*

If the functions $f_h(\xi)$ $(h=0, 1, \ldots)$ are defined on N', then it is possible to find a neutrix U with domain N' such that each function negligible in N is negligible in U and that each of the functions $f_0(\xi)$, $f_1(\xi), \ldots$ assumes at U a neutralized value.

Proof: According to the preceding theorem we can construct a neutrix N_0 such that $f_0(\xi)$ assumes at N_0 a neutralized value, that each function negligible in N is negligible in N_0 and that each function $\nu(\xi)$ negligible in N_0 and each rational number ρ have the property that $\rho\nu(\xi)$ is negligible in N_0. In the same way we construct a neutrix N_1 such that $f_1(\xi)$ assumes at N_1 a neutralized value and that each function negligible in N_0 is negligible in N_1, and so on. Continuing in this way we construct infinitely many neutrices N_0, N_1, ... such that $f_h(\xi)$ $(h=0, 1, \ldots)$ assumes a neutralized value

at N_h, and that each function negligible in N_h is also negligible in N_{h+1}. The union U of these neutrices $N_0, N_1, \ldots$ possesses according to Theorem 1.3.1 the required property.

Remark: This theorem has the following meaning. Assume the functions $f_h(\xi)$ $(h=0, 1, \ldots)$ are defined on a non-empty set N'. According to Theorem 1.3.3 there exists a neutrix U with domain N' which neutralizes all these functions. This neutrix leads to a calculus in which all the functions belonging to U may be neglected. Suppose that later we meet with other functions $g_h(\xi)$ $(h=0, 1, \ldots)$ defined on N'. We do not know whether these new functions are neutralized by U but by means of the preceding theorem we can construct a neutrix V with domain N' and with the property that each function negligible in U is also negligible in V and that each new function $g_h(\xi)$ assumes at V a neutralized value. Each old function $f_h(\xi)$ assumes at U a neutralized value γ_h and can therefore be written in the form

$$f_h(\xi) = \gamma_h + \nu_h(\xi),$$

where $\nu_h(\xi)$ is negligible in U and therefore certainly in V. Consequently each old function $f_h(\xi)$ is neutralized by the new neutrix V and assumes at V the same neutralized value as at U. This means that each formula valid in the calculus based on the neutrix U remains valid in the new calculus, if we replace in this formula U by V. Moreover, if the new neutrix V is chosen in an appropriate way, then in the new calculus we may neglect functions which may not be neglected in the calculus based on U. Such an enlargement may lead to great simplifications. Whenever we meet with new functions we can always enlarge our neutrix in such a way that these functions are neutralized by the enlarged neutrix.

Theorem 1.3.4: *Let M and N be two neutrices such that the domain M' of M is a subset of the domain N' of N and that each function $\nu(\xi)$ defined on N' and negligible in N has the property that the corresponding function $\nu(\eta)$ defined at each element η of M' is negligible in M. Each function $g(\xi)$ defined on N' which assumes at N a neutralized value has the property that the corresponding function $g(\eta)$ defined on M' assumes at M the same neutralized value.*

Proof: We have for each element ξ of N'

$$g(\xi) = \gamma + \nu(\xi),$$

where $\gamma = g(N)$ is independent of ξ and where $\nu(\xi)$ is negligible in N. Consequently for each element η of M'

$$g(\eta) = \gamma + \nu(\eta)$$

where $\nu(\eta)$ is negligible in M, so that $g(\eta)$ assumes at M the neutralized value γ.

1.4. Integrating neutrices.

Definition: Assume $a < b$; a may be $-\infty$ and b may be ∞. We say that a function is integrable from $a+$ to b if it is integrable from α to b for each point α lying between a and b. We say that a function is integrable from a to $b-$ if it is integrable from a to β for each point β lying between a and b. We say that a function is integrable from $a+$ to $b-$ if it is integrable from α to β for any two points α and β with $a < \alpha < \beta < b$.

An integrating neutrix I_{a+} with domain $a < \xi < b$ is a neutrix with the two following properties:

(1) for each function $g(x)$ integrable from a to $b-$ the function

$$\int_a^\xi g(x)\,dx \qquad (a < \xi < b) \tag{1.4.1}$$

is negligible in I_{a+}.

(2) each function $\nu(\xi)$ negligible in I_{a+} and each rational number ρ have the property that $\rho\nu(\xi)$ is negligible in I_{a+}.

That such a neutrix exists follows from the following theorem.

Theorem 1.4.1: Let N be a neutrix with domain $a < \xi < b$ with the two following properties:

(1) each function defined in $a < \xi < b$ and tending to zero for $\xi \to a$ is negligible in N,

(2) each function $\nu(\xi)$ negligible in N and each rational number ρ have the property that $\rho\nu(\xi)$ is negligible in N.

Then N is an integrating neutrix I_{a+}.

Proof: The integral (1.4.1) tends to zero for $\xi \to a$ and is therefore negligible in N.

Let I_{a+} be an arbitrary integrating neutrix with domain $a < \xi < b$. Assume $\beta \geqq b$. We say that a function integrable from $a+$ to β is integrable from I_{a+} to β if the function

$$\int_{\xi}^{\beta} f(x)\, dx \qquad (a < \xi < b)$$

assumes at I_{a+} a neutralized value. According to the convention made in the preceding section this neutralized value is denoted by

$$\int_{I_{a+}}^{\beta} f(x)\, dx\,.$$

Example: All the functions of the form

$$c\xi^{-1} + o(1) \qquad (0 < \xi < 1)$$

where $o(1)$ denote functions of ξ which tend to zero for $\xi \to 0$ and where the coefficients c are arbitrary complex constants, form an integrating neutrix I_{0+} with domain $0 < \xi < 1$. That the neutrix condition is satisfied is obvious, for if

$$c\xi^{-1} + o(1) = \gamma \qquad (0 < \xi < 1)$$

where γ is independent of ξ, then $c = 0$, hence $\gamma = 0$. This neutrix has the property

$$\int_{I_{0+}}^{1} x^{-2}\, dx = -1$$

since

$$\int_{\xi}^{1} x^{-2}\, dx = \xi^{-1} - 1$$

where ξ^{-1} is negligible in I_{0+}.

Theorem 1.4.2: *If I_{a+} is an integrating neutrix with domain $a < \xi < b$ and if $f(x)$ is integrable from a to β, where $\beta \geqq b$, then $f(x)$ is certainly integrable from I_{a+} to β and we have*

$$\int_{I_{a+}}^{\beta} f(x)\,dx = \int_{a}^{\beta} f(x)\,dx\,.$$

Proof: By definition the integral

$$\int_{a}^{\xi} f(x)\,dx \qquad (a<\xi<b)$$

is negligible in I_{a+}, so that

$$\int_{\xi}^{\beta} f(x)\,dx = \int_{a}^{\beta} f(x)\,dx - \int_{a}^{\xi} f(x)\,dx \tag{1.4.2}$$

assumes at I_{a+} a neutralized value which is equal to the first term occurring on the right-hand side of (1.4.2).

Theorem 1.4.3: If $a<\beta$, if $f(x)$ and $g(x)$ are integrable from I_{a+} to β and if ρ is a rational constant, then $f(x)+g(x)$, $f(x)-g(x)$ and $\rho f(x)$ are integrable from I_{a+} to β and we have

$$\int_{I_{a+}}^{\beta} (f(x)+g(x))\,dx = \int_{I_{a+}}^{\beta} f(x)\,dx + \int_{I_{a+}}^{\beta} g(x)\,dx$$

$$\int_{I_{a+}}^{\beta} (f(x)-g(x))\,dx = \int_{I_{a+}}^{\beta} f(x)\,dx - \int_{I_{a+}}^{\beta} g(x)\,dx$$

and

$$\int_{I_{a+}}^{\beta} \rho f(x)\,dx = \rho \int_{I_{a+}}^{\beta} f(x)\,dx\,.$$

This is obvious.

Theorem 1.4.4: Assume $a<b$. If infinitely many functions $f_h(x)$ $(h=1,2,\ldots)$ are integrable from $a+$ to b, then there exists an integrating neutrix I_{a+} with domain $a<\xi<b$ such that each of these functions $f_h(x)$ $(h=1,2,\ldots)$ is integrable from I_{a+} to b.

Proof: Let I^*_{a+} be an arbitrary integrating neutrix. According to Theorem 1.3.3 there exists a neutrix U such that each function negligible

in I^*_a is negligible in U, that a rational number multiplied by a function negligible in U is again negligible in U and that each function

$$\int_{\xi}^{b} f_h(x)\,dx \qquad (a<\xi<b)$$

$(h=1, 2, \ldots)$ assumes at U a neutralized value. U is an integrating neutrix I_{a+} with the required properties. Indeed, if $g(x)$ is integrable from a to b—, then

$$\int_{a}^{\xi} g(x)\,dx \qquad (a<\xi<b)$$

is negligible in I^*_{a+}, therefore certainly in U.

Definition: An integrating neutrix I_{b-} with domain $a<\xi<b$ is a neutrix with the two following properties:

(1) for each function $g(x)$ integrable from $a+$ to b the function

$$\int_{\xi}^{b} g(x)\,dx \qquad (a<\xi<b)$$

is negligible in I_{b-},

(2) each function $\nu(\xi)$ negligible in I_{b-} and each rational constant ρ have the property that $\rho\nu(\xi)$ is negligible in I_{b-}.

Assume $\alpha \leqq a$. We say that a function integrable from α to $b-$ is integrable from α to I_{b-} if the function

$$\int_{\alpha}^{\xi} f(x)\,dx \qquad (a<\xi<b)$$

assumes at I_{b-} a neutralized value; this value is denoted by

$$\int_{\alpha}^{I_{b-}} f(x)\,dx.$$

The integrals with limit I_{b-} have properties similar to those of the integrals with limit I_{a+}. These properties are so simple that it is not necessary to formulate them.

If $a<c<b$ and if $f(x)$ is integrable from I_{a+} to c and also from c to I_{b-}, then $f(x)$ is integrable from I_{a+} to I_{b-} and we have

$$\int_{I_{a+}}^{I_{b-}} f(x)\,dx = \int_{I_{a+}}^{c} f(x)\,dx + \int_{c}^{I_{b-}} f(x)\,dx\,.$$

Notice that the value of the right-hand side is independent of the choice of the point c lying between a and b.

1.5. The periodic neutrix.

Definition: A smooth function (smooth at ∞) is a function $f(x)$ defined for sufficiently large positive x with the property that there exists a positive constant integer m such that $f(x)$ is m times continuously differentiable, that

$$f^{(m)}(x) \to 0 \qquad \text{as } x \to \infty \tag{1.5.1}$$

and that $f^{(m)}(x)$ is absolutely integrable to infinity.

The periodic neutrix P with domain $\alpha < \xi < \infty$ is the class formed by all the functions $\nu(\xi)$ defined in the interval $\alpha < \xi < \infty$ which can be written in the form

$$\nu(\xi) = \sum_{h=0}^{n-1} s_h(\xi)\, p_h(\xi) + o(1) \qquad (\alpha < \xi < \infty) \tag{1.5.2}$$

where $o(1)$ denotes a function of ξ which tends to zero as $\xi \to \infty$, where n denotes an integer $\geqq 0$ independent of ξ, where $s_h(\xi)$ $(h = 0, 1, \dots, n-1)$ denote smooth functions and where $p_h(\xi)$ $(h = 0, 1, \dots, n-1)$ denote periodic, bounded, integrable functions with period 1 and with

$$\int_0^1 p_h(x)\,dx = 0\,. \tag{1.5.3}$$

In Section 7.1, I shall show that P satisfies the neutrix condition.

Theorem 1.5.1: *If $a \leqq \alpha$, if I_{a+} is an integrating neutrix and if the smooth function $f(x)$ defined for $x > a$ is integrable from I_{a+} to $\infty-$, then*

$$r(I_{a+}, \xi, f) = \sum_{a<n<\xi} f(n) - \int_{I_{a+}}^{\xi} f(x)\,dx \tag{1.5.4}$$

is a function of $\xi > \alpha$ which assumes a neutralized value

at the periodic neutrix P with domain $\alpha < \xi < \infty$. This neutralized value is called the residue $r(I_{a+}, P, f)$ at I_{a+} of f with neutrix P. In the particular case that $f(x)$ is integrable from a to $\infty -$, then (1.5.4) is independent of the choice of the neutrix I_{a+} and then I denote $r(I_{a+}, \xi, f)$ and $r(I_{a+}, P, f)$ simply by $r(a, \xi, f)$ and $r(a, P, f)$.

Each positive integer m with (1.5.1) and with the property that $f^{(m)}(x)$ is absolutely integrable to infinity satisfies for each sufficiently large ξ, the relation

$$(1.5.5) \quad r(I_{a+}, P, f) = r(I_{a+}, \xi, f) - \Lambda_m(\xi -, f) + (-)^{m-1} \int_{\xi}^{\infty} f^{(m)}(x) \varphi_m(x) dx.$$

Proof: Choose $\alpha \geqq a$ in such a way that $f(x)$ is m times continuously differentiable for $x \geqq \alpha$. If $\alpha \leqq \xi^* \leqq \xi$, then

$$r(I_{a+}, \xi, f) - r(I_{a+}, \xi^*, f) = \sum_{\xi^* \leqq n < \xi} f(n) - \int_{\xi^*}^{\xi} f(x) dx$$

is according to the primitive sum formula (1.2.1) equal to

$$(1.5.6) \quad r(I_{a+}, \xi, f) - r(I_{a+}, \xi^*, f)$$

$$= \Lambda_m(\xi -, f) - \Lambda_m(\xi^* -, f) + (-)^{m-1} \int_{\xi^*}^{\xi} f^{(m)}(x) \varphi_m(x) dx.$$

Interchanging ξ and ξ^* we see that this formula is also true if $\alpha \leqq \xi \leqq \xi^*$. The integrand occurring in (1.5.6) is integrable to infinity, so that

$$(1.5.7) \quad r(I_{a+}, \xi, f) = r(I_{a+}, \xi^*, f) - \Lambda_m(\xi^* -, f) + (-)^{m-1} \int_{\xi^*}^{\infty} f^{(m)}(x) \varphi_m(x) dx$$

$$+ \Lambda_m(\xi -, f) - (-)^{m-1} \int_{\xi}^{\infty} f^{(m)}(x) \varphi_m(x) dx.$$

The last term tends to zero as $\xi \to \infty$. Each term of the sum $\Lambda_m(\xi -, f)$ has according to the definition given in (1.2.2) the form $s(\xi) p(\xi)$, where $s(\xi)$ is smooth and where $p(\xi)$ is periodic with the period 1, bounded and integrable with

$$\int_0^1 p(x)\,dx = 0$$

according to (1.2.3). Consequently $r(I_{a+}, P, f)$ exists and is equal to the sum of the three first terms occurring in the right-hand side of (1.5.7). This completes the proof.

Remark:

$$r(I_{a+}, P, f) = \sum_{n>a} f(n) - \int_{I_{a+}} f(x)\,dx \tag{1.5.8}$$

if both the sum and the integral occurring on the right-hand side converge. Indeed the difference between the right-hand sides of (1.5.4) and (1.5.8) tends to zero for $\xi \to \infty$ and is therefore negligible in P.

Theorem 1.5.2 (theorem of translation): If u is an integer and $r(I_{a+u+}, P, f(x-u))$ exists, then $r(I_{a+}, P, f(x))$ exists and has the same value.

Proof: For each $\eta > a + u$ we have

$$\begin{aligned} r(I_{a+u+}, \eta, f(x-u)) &= \sum_{a+u<n<\eta} f(n-u) - \int_{I_{a+u+}}^{\eta} f(x-u)\,dx \\ &= \sum_{a<n<\eta-u} f(n) - \int_{I_{a+}}^{\eta-u} f(x)\,dx \\ &= r(I_{a+}, \eta - u, f(x)). \end{aligned}$$

By hypothesis

$$r(I_{a+u+}, \eta, f(x-u)) = \gamma + \nu(\eta)$$

where $\gamma = r(I_{a+u+}, P, f(x-u))$ and where $\nu(\eta)$ is negligible in P. Consequently for $\xi > a$

$$r(I_{a+}, \xi, f(x)) = \gamma + \nu(\xi+u). \tag{1.5.9}$$

If $s(x)$ is smooth, then $s(x+u)$ is also smooth. From the definition of the periodic neutrix P it follows therefore that $\nu(\xi+u)$ is for each choice of the integer u negligible in P. Formula (1.5.9) shows therefore that $r(I_{a+}, P, f(x))$ exists and is equal to γ. This establishes the proof.

1.6. The periodic neutrix $-P$.

Definition: If $f(x)$ is smooth at ∞, then $f(-x)$ is by definition smooth at $-\infty$. If P is the periodic neutrix with domain $\alpha<\xi<\infty$, then the functions $f(x)$ defined for $x>\alpha$ and belonging to P have the property that the functions $f(-x)$ form a neutrix with domain $-\infty<\eta<-\alpha$; this neutrix is called the periodic neutrix $-P$ with domain $-\infty<\eta<-\alpha$.

Theorem 1.5.1 gives

Theorem 1.6.1: If $\beta\leqq b$ and if a function $f(x)$ defined for $-\infty<x<b$ and smooth at $-\infty$ is integrable from $-\infty+$ to an integrating neutrix I_{b-}, then

$$r(I_{b-},\xi,f)=\sum_{\xi<n<b} f(n)-\int_{\xi}^{I_{b-}} f(x)\,dx$$

is a function of $\xi<\beta$ which assumes a neutralized value at the periodic neutrix $-P$ with domain $-\infty<x<\beta$.

This neutralized value is called the residue $r(I_{b-},-P,f)$ at I_{b-} of f with neutrix $-P$.

Each positive integer m such that $f^{(m)}(x)\to 0$ as $x\to-\infty$ and that $f^{(m)}(x)$ is absolutely integrable from $-\infty$ satisfies for each $\xi<0$ with sufficiently large $|\xi|$, the relation

$$r(I_{b-},-P,f)=r(I_{b-},\xi,f)+\Lambda_m(\xi+,f)+(-)^{m-1}\int_{-\infty}^{\xi} f^{(m)}(x)\,\varphi_m(x)\,dx\,.$$

Theorem 1.6.2 (theorem of translation): If u is an integer and $r(I_{u-b+},P,f(u-x))$ exists, then $r(I_{b-},-P,f(x))$ exists and has the same value.

This follows from Theorem 1.5.2.

CHAPTER 2. ASYMPTOTIC NEUTRICES

2.1. Definition of asymptotic neutrices. In asymptotics we consider an unbounded point set Ω lying in the complex plane or on a Riemann surface; the elements of Ω are denoted by ω. The purpose of asymptotics is to determine for the elements ω of Ω with large $|\omega|$ the asymptotic behavior of certain functions of ω. The numbers, functions, sets and so on

occurring in this paper may always depend on ω unless they are called fixed.

We call a and b asymptotically equal and we write

$$a \sim b \tag{2.1.1}$$

if for each fixed real q they satisfy the order relation

$$a - b = O\,|\omega|^{-q}\,. \tag{2.1.2}$$

This means that for each fixed real q it is possible to find two fixed numbers c and γ such that for each point ω of Ω with $|\omega| \geqq \gamma$ the numbers a and b are defined and satisfy the inequality

$$|a - b| \leqq c\,|\omega|^{-q}\,. \tag{2.1.3}$$

It may be that a and b depend not only on ω, but also on other parameters s, t, ... If for each fixed q the numbers c and γ with the said properties can be chosen independent of s, t, ... , then we say that the relations (2.1.1) and (2.1.2) hold uniformly in s, t, ...

For instance, if Ω is formed by the numbers $\omega \geqq 1$, then we have for each fixed number $s \neq 0$

$$s^{-1}\,e^{-\omega} \sim 0$$

but this relation is not uniform in s.

Definition: Let N' be a non-empty set and let N be an additive group formed by functions $\nu(\xi)$ defined on N'. This group is called an asymptotic neutrix when it satisfies the

Asymptotic neutrix condition: If γ is independent of ξ and if for each real q it is possible to find in the additive group N a function $\nu(\xi)$ such that the order relation

$$\nu(\xi) = \gamma + O\,|\omega|^{-q}$$

holds for each element ξ of N', uniformly in ξ, then $\gamma \sim 0$.

Example 1: Let s be a fixed number $\neq 0$. If Ω is formed by the numbers $\omega \geqq 1$, then the functions $c\,\xi\,s^{-1}\,e^{-\omega}$, where the coefficients c denote arbitrary fixed integers, form an asymptotic neutrix with domain $1 < \xi < 2$. Indeed if a number γ which may depend on ω and s but not on ξ and q has the property that for each real q it is possible to find an integer c, which may depend on s and q but not on ω and ξ such that

$$\gamma = c\,\xi\,s^{-1}e^{-\omega} + O\,\omega^{-q} \qquad (1 < \xi < 2)$$

then $\gamma = O\,\omega^{-q}$, hence $\gamma \sim 0$. The relations $\gamma = O\,\omega^{-q}$ and $\gamma \sim 0$ are not necessarily uniform in s.

Definition: Let $g(\xi)$ be defined for each element ξ of the domain N' of an asymptotic neutrix N. If γ, independent of ξ, has the property that for each fixed real q it is possible to find a function $\nu(\xi)$ negligible in N such that the order relation

$$g(\xi) = \gamma + \nu(\xi) + O\,\omega^{-q}$$

holds for each element ξ of N', uniformly in ξ, then the asymptotic behavior of γ is uniquely defined. Indeed if we can write $g(\xi)$ also in the form

$$g(\xi) = \gamma_1 + \nu_1(\xi) + O\,|\omega|^{-q}$$

where γ_1 is independent of ξ and q and where $\nu_1(\xi)$ is negligible in N, then

$$\gamma - \gamma_1 = \nu_1(\xi) - \nu(\xi) + O\,|\omega|^{-q}$$

where $\nu_1(\xi) - \nu(\xi)$ is negligible in N, so that $\gamma - \gamma_1 \sim 0$ according to the neutrix condition satisfied by N.

We say that γ is the neutralized asymptotic value which $g(\xi)$ assumes at the asymptotic neutrix N. This value is denoted by $g(N)$. Notice that $g(N)$ is not uniquely defined; only its asymptotic behavior is uniquely defined.

Example 2: Let Ω be the set formed by the numbers $\omega \geqq 1$. Let N' be the interval $\omega < \xi < 2\omega$ and let N be the asymptotic neutrix formed by the polynomials in ξ with fixed degree and without constant term; the coefficients in these polynomials may depend on ω. Then

$$\sqrt{\omega^2 + N} \sim \omega. \tag{2.1.4}$$

Proof: We must first show that N satisfies the asymptotic neutrix condition. Let γ be a number independent of ξ. Assume that for each fixed real q it is possible to find a polynomial $\sum_{h=1}^{n} a_h \xi^h$ such that the order relation

$$\gamma = \sum_{h=1}^{n} a_h \xi^h + O\,\omega^{-q}$$

holds uniformly in ξ $(\omega < \xi < 2\omega)$; the degree n may depend on q but not on ω; the coefficients a_h may depend on ω and q. We must prove then that $\gamma \sim 0$. If we choose in the interval $1 < p < 2$, $n+1$ distinct fixed numbers $p_0, p_1, \ldots, p_n$, then

$$\sum_{h=1}^{n} a_h \omega^h p_k^h = \gamma + O\,\omega^{-q} \qquad (k = 0, 1, \dots, n). \tag{2.1.5}$$

The determinant v of Vandermonde formed by the $(n+1)^2$ elements p_k^h $(h = 0, 1, \dots, n;\ k = 0, 1, \dots, n)$ is a fixed number $\neq 0$. Multiplying both sides of (2.1.5) with the minor of p_k^0 in this determinant and adding we obtain

$$0 = \gamma v + O\,\omega^{-q}, \qquad \text{hence} \quad \gamma = O\,\omega^{-q}$$

consequently $\gamma \sim 0$, so that N satisfies the asymptotic neutrix condition.

In the interval $\omega < \xi < 2\omega$ we have for each fixed q and for each fixed integer $l \geqq q + 1$

$$\sqrt{\omega^2 + \xi} = \omega + \sum_{h=1}^{l-1} \binom{\frac{1}{2}}{h} \omega^{1-2h} \xi^h + O\,\omega^{-q}$$

where the sum $\sum\limits_{h=1}^{l-1}$ is negligible in N. This gives the required result (2.1.4).

Theorem 2.1.1: Let M and N be two asymptotic neutrices such that the domain M' of M is a subset of the domain N' of N and that each function $\nu(\xi)$ defined on N' and negligible in N has the property that the corresponding function $\nu(\eta)$ defined on M' is negligible in M. Each function $g(\xi)$ defined on N' which assumes at N a neutralized value has the property that the corresponding function $g(\eta)$ defined on M' assumes at M a neutralized value and these two neutralized values are asymptotically equal.

Proof: For each fixed real q we can find a function $\nu(\xi)$ negligible in N such that the order relation

$$g(\xi) = \gamma + \nu(\xi) + O\,|\omega|^{-q}$$

holds, uniformly in ξ, for each element ξ of N'; here $\gamma \sim g(N)$.
Then

$$g(\eta) = \gamma + \nu(\eta) + O\,|\omega|^{-q}$$

uniformly in η, for each element η of M', so that $g(M)$ exists and is asymptotically equal to γ. This establishes the proof.

2.2. The periodic asymptotic neutrix.

Definition: Assume $\alpha < \beta$; α may be $-\infty$ and β may be ∞.

A function $f(x)$ is called asymptotically smooth in $\alpha < x < \beta$ if it is, in this interval, infinitely often differentiable; if, for large $|\omega|$, the derivatives $f^{(m)}(x)$ tend, uniformly in x $(\alpha < x < \beta)$, asymptotically to zero as $m \to \infty$; and if finally

$$\int_{\alpha}^{\beta} | f^{(m)}(x) | \, dx$$

tends asymptotically to zero as $m \to \infty$.

Assume that $\beta - \alpha \to \infty$ as $|\omega| \to \infty$. The periodic asymptotic neutrix Q with domain $\alpha < \xi < \beta$ is the class formed by the functions $\pi(\xi)$ $(\alpha < \xi < \beta)$ which for each real fixed q satisfy an order relation of the form

$$\pi(\xi) = \sum_{h=0}^{n-1} s_h(\xi)\, p_h(\xi) + O\,|\omega|^{-q} \tag{2.2.1}$$

uniformly in ξ $(\alpha < \xi < \beta)$, where n is independent of ω and ξ, where the functions $s_h(\xi)$ $(h = 0, 1, \ldots, n-1)$ are asymptotically smooth in $\alpha < \xi < \beta$ and where the functions $p_h(\xi)$ $(h = 0, 1, \ldots, n-1)$ are periodic, bounded, integrable functions with period 1 and

$$\int_0^1 p_h(x)\, dx = 0 . \tag{2.2.2}$$

Bounded means here: in absolute value less than a suitably chosen positive number independent of ω and ξ; the integer n may depend on q; the functions $s_h(\xi)$ and $p_h(\xi)$ $(h = 0, 1, \ldots, n-1)$ may depend on ω and q.

Remark: (1) We say that the numbers a_m $(m = 0, 1, \ldots)$ which may depend on ω tend for $m \to \infty$ asymptotically to zero if it is possible to find for each fixed real q an integer $m_0 \geqq 0$ such that each fixed integer $m \geqq m_0$ satisfies for large $|\omega|$ the order relation

$$a_m = O\,|\omega|^{-q} .$$

The main theorem [1] in the theory of asymptotic series can be formulated as follows;

1) J. G. van der Corput, Asymptotic Developments I. Fundamental theorems of asymptotics, Journal d'Analyse Mathématique 4, 1954-56, p. 341-418; see page 367, Theorem 4.1.

If in a series $\sum_{m=0}^{\infty}{}' a_m$ the term a_m tends asymptotically to zero for $m \to \infty$, then the series converges asymptotically; this means that there exists a function s of ω such that

$$s - \sum_{h=0}^{m-1} a_h$$

tends, for $m \to \infty$, asymptotically to zero. The function s is called the asymptotic sum of the series. This function is not uniquely defined by the series, only its asymptotic behavior is uniquely determined.

(2) The statement that, for large $|\omega|$, certain functions $f_m(x)$ $(m = 0, 1, \ldots)$ tend, uniformly in x $(\alpha < x < \beta)$, asymptotically to zero as $m \to \infty$ has the following meaning: for each fixed real q, it is possible to find a fixed integer $m_0 \geqq 0$ such that each fixed integer $m \geqq m_0$ satisfies for large $|\omega|$ the order relation

$$f_m(x) = O\,|\omega|^{-q}$$

uniformly in $x\,(\alpha < x < \beta)$.

(3) If the functions $f_m(x)$ $(m = 0, 1, \ldots)$ are defined for sufficiently large positive x, then the statement that, for large positive x, the functions $f_m(x)$ tend asymptotically to zero as $m \to \infty$ has the following meaning: for each fixed real q it is possible to find a fixed integer $m_0 \geqq 0$ such that each fixed integer $m \geqq m_0$ satisfies for large positive x the order relation

$$f_m(x) = O\,x^{-q}.$$

Theorem 2.2.1: If $a \leqq \alpha < \beta$, where $\beta - \alpha \to \infty$ as $|\omega| \to \infty$, and if $f(x)$ defined for $a < x < \beta$, is integrable from an integrable neutrix I_{a+} to $\beta-$ and if finally $f(x)$ is asymptotically smooth in $\alpha < x < \beta$, then

$$r(I_{a+}, \xi, f) = \sum_{a<n<\xi} f(n) - \int_{I_{a+}}^{\xi} f(x)\,dx \tag{2.2.3}$$

is a function of ξ $(\alpha < \xi < \beta)$ which assumes a neutralized value at the periodic asymptotic neutrix Q with domain $\alpha < \xi < \beta$.

This neutralized value is called the residue $r(I_{a+}, Q, f)$ at I_{a+} of f with the neutrix Q and possesses, uniformly in ξ $(\alpha<\xi<\beta)$ the asymptotic expansion

$$r(I_{a+}, Q, f) \sim r(I_{a+}, \xi, f) - \Lambda_\infty(\xi-, f), \tag{2.2.4}$$

where $\Lambda_\infty(\xi-, f)$ denotes the asymptotic sum of the asymptotically convergent series

$$\Lambda_\infty(\xi-, f) \sim \sum_{h=0}^{\infty}{}' (-)^{h+1} f^{(h)}(\xi)\, \varphi_{h+1}(\xi-).$$

In the special case that $f(x)$ is integrable from a to $\beta-$, then $r(I_{a+}, \xi, f)$ and $r(I_{a+}, Q, f)$ are independent of the choice of the integrating neutrix I_{a+} and are denoted simply by $r(a, \xi, f)$ and $r(a, Q, f)$.

Proof: Formula (1.5.6) holds for any two points ξ and ξ^* lying in the interval $\alpha<x<\beta$. If q is a fixed real number, then we have, for sufficiently large fixed integer m,

$$\int_{\xi^*}^{\xi} f^{(m)}(x)\, \varphi_m(x)\, dx = O\, |\omega|^{-q} \quad \text{and} \quad \Lambda_m(\xi^*-, f) = \Lambda_\infty(\xi^*-, f) + O\, |\omega|^{-q}$$

so that

$$r(I_{a+}, \xi, f) = r(I_{a+}, \xi^*, f) - \Lambda_\infty(\xi^*-, f) + \Lambda_m(\xi-, f) + O\, |\omega|^{-q}. \tag{2.2.5}$$

Each term in the sum $\Lambda_m(\xi-, f)$ has the form $s(\xi)\, p(\xi)$, where $s(\xi)$ is asymptotically smooth in $\alpha<\xi<\beta$ and where $p(\xi)$ is periodic, bounded and integrable with

$$\int_0^1 p(x)\, dx = 0 \tag{2.2.6}$$

by (1.2.3). Consequently each term occurring in the sum $\Lambda_m(\xi-, f)$ is negligible in Q, so that $r(I_{a+}, Q, f)$ exists and is asymptoticically equal to the sum of the first two terms occurring on the right hand side of (2.2.5). This completes the proof.

Theorem 2.2.2: If the conditions of the preceding theorem are satisfied and if the function $f(\xi)$ and each of

its derivatives tend to zero as $\xi < \beta$ tends to β, then

$$r(I_{a+}, Q, f) \sim \lim_{\xi \to \beta} r(I_{a+}, \xi, f)$$

provided that the limit occurring on the right-hand side exists.

Proof: According to (2.2.4) there exists for each fixed real q a positive integer m independent of ω and ξ such that in the interval $\alpha < \xi < \beta$, uniformly in ξ

$$r(I_{a+}, Q, f) = r(I_{a+}, \xi, f) - \Lambda_m(\xi-, f) + O\,|\omega|^{-q}.$$

As $\xi \to \beta$ each term in the sum $\Lambda_m(\xi-, f)$ tends by hypothesis to zero, so that

$$r(I_{a+}, Q, f) = \lim_{\xi \to \beta} r(I_{a+}, \xi, f) + O\,|\omega|^{-q}.$$

This holds for each fixed real q; this gives the required result.

Theorem 2.2.3: Assume $a \leqq \alpha$. Assume that $f(x)$ defined for $x > a$ is integrable from an integrating neutrix I_{a+} to $\infty-$. If $f(x)$ is asymptotically smooth in the interval $\alpha < x < \infty$ and if, for each fixed sufficiently large positive integer m,

$$f^{(m)}(x) \to 0 \qquad \text{as} \quad x \to \infty, \tag{2.2.7}$$

then $r(I_{a+}, Q, f)$ and $r(I_{a+}, P, f)$ exist and

$$r(I_{a+}, Q, f) \sim r(I_{a+}, P, f); \tag{2.2.8}$$

here Q is the periodic asymptotic neutrix with domain $\alpha < \xi < \infty$.

Remark: This theorem forms the bridge between the residues with neutrix Q and the — in many respects simpler — residues with neutrix P.

Proof: Applying Theorem 1.5.1 and using the fact that for $\xi > \alpha$ the last term occurring in (1.5.5) tends asymptotically to zero as $m \to \infty$ we obtain

$$r(I_{a+}, P, f) \sim r(I_{a+}, \xi, f) - \Lambda_\infty(\xi-, f)$$

and in Theorem 2.2.1 we have found the same asymptotic expansion for $r(I_{a+}, Q, f)$.

Theorem 2.2.4: If $\beta - \alpha$ is a posivive number tending to infinity as $|\omega| \to \infty$ and if $f(x)$ is asymptotically smooth in

the interval $a \leqq x < \beta$, then

$$r(a, Q, f) \sim -\Lambda_\infty(a+, f), \tag{2.2.9}$$

where Q is the periodic asymptotic neutrix with domain $a < x < \beta$ and where the right-hand side denotes the asymptotic sum of the asymptotically convergent series

$$-\sum_{h=0}^{\infty}{}' (-)^{h+1} f^{(h)}(a) \varphi_{h+1}(a+).$$

Proof: According to the primitive sum formula (1.2.1) of Euler we have for $a < \xi < \beta$

$$r(a, \xi, f) = \Lambda_m(\xi-, f) - \Lambda_m(a+, f) + (-)^{m-1} \int_a^\xi f^{(m)}(x) \varphi_m(x)\, dx.$$

If q is a given real number, then the last term is for sufficiently large fixed integer m, certainly $O|\omega|^{-q}$ uniformly in ξ. Each term in the sum $\Lambda_m(\xi-, f)$ is negligible in Q, so that for sufficiently large fixed integer m

$$r(a, Q, f) = -\Lambda_m(a+, f) + O|\omega|^{-q}.$$

If we choose this integer m sufficiently large, then

$$\Lambda_m(a+, f) = \Lambda_\infty(a+, f) + O|\omega|^{-q}$$

hence

$$r(a, Q, f) = -\Lambda_\infty(a+, f) + O|\omega|^{-q}.$$

This holds for each fixed real number q and yields therefore the required result.

2.3. Asymptotic expansions for residues.

Theorem 2.3.1: If N is an asymptotic neutrix with domain N' and if for each element ξ of N' uniformly in ξ,

$$g(\xi) \sim \sum_{h=0}^{\infty}{}' g_h(\xi)$$

with the property that $g_h(N)$ $(h=0, 1, \ldots)$ exists and tends asymptotically to zero for $h \to \infty$, then $g(N)$ exists and

$$g(N) \sim \sum_{h=0}^{\infty}{}' g_h(N) . \tag{2.3.1}$$

Proof: For each fixed real q we have, if the positive fixed integer m is sufficiently large,

$$g(\xi) = \sum_{h=0}^{m-1} g_h(\xi) + O\,|\omega|^{-q}$$

uniformly in ξ for each element ξ of N', hence

$$g(\xi) = \sum_{h=0}^{m-1} g_h(N) + \nu(\xi) + O\,|\omega|^{-q} \sim \sum_{h=0}^{\infty}{}' g_n(N) + \nu(\xi) + O\,|\omega|^{-q}$$

where $\nu(\xi)$ is negligible in N. This holds for each fixed real q, so that $g(N)$ exists and satisfies (2.3.1).

Definition: A number u is called asymptotically finite if there exists a fixed real number p such that

$$u = O\,|\omega|^{p} . \tag{2.3.2}$$

Theorem 2.3.2: Let N be an asymptotic neutrix such that each function $\nu(\xi)$ negligible in N and each asymptotically finite number u has the property that $u\nu(\xi)$ is negligible in N (for instance each periodic asymptotic neutrix Q satisfies this condition). If $g(\xi)$ assumes at N a neutralized value $g(N)$, then each asymptotically finite number u has the property that $ug(\xi)$ assumes at N a neutralized value which is asymptotically equal to $ug(N)$.

Proof: Let p denote a fixed real number with property (2.3.2.) For each fixed real q we can find a function $\nu(\xi)$ negligible in N such that the order relation

$$g(\xi) = \gamma + \nu(\xi) + O\,|\omega|^{-q-p}$$

holds uniformly in ξ, where $\gamma = g(N)$. Consequently

$$u\,g(\xi) = u\gamma + u\nu(\xi) + O\,|\omega|^{-q}$$

where $u\nu(\xi)$ is negligible in N, so that $ug(\xi)$ assumes at N a neutralized value which is asymptotically equal to $u\,\gamma$.

Theorem 2.3.3: Assume $a \leqq \alpha < \beta$, where $\beta - \alpha \to \infty$ as $|\omega| \to \infty$. Let Q be the periodic asymptotic neutrix with domain $\alpha < \xi < \beta$. Let u_h ($h = 0, 1, \ldots$) be a number independent of x which is, for each fixed integer $h \geqq 0$, asymptotically finite. If the functions $g_h(x)$ ($h = 0, 1, \ldots$) defined for $x > a$ are integrable from an integrating neutrix I_{a+} to $\beta-$, if $r(I_{a+}, Q, g_h)$ ($h = 0, 1, \ldots$), exists and $u_h r(I_{a+}, Q, g_h)$ tends asymptotically to zero as $h \to \infty$, then each function $f(x)$ integrable from I_{a+} to $\beta-$ such that

$$r\left(I_{a+}, \xi, f - \sum_{h=0}^{m-1} u_h g_h\right)$$

tends asymptotically to zero in the interval $a < \xi < \beta$, uniformly in ξ, as $m \to \infty$, has the property that $r(I_{a+}, Q, f)$ exists and possesses the asymptotic expansion

$$r(I_{a+}, Q, f) \sim \sum_{h=0}^{\infty}{}' u_h r(I_{a+}, Q, g_h).$$

Proof: Put

$$f(x) = \sum_{h=0}^{m-1} u_h g_h(x) + \rho_m(x)$$

so that in the interval $\alpha < \xi < \beta$

$$r(I_{a+}, \xi, f) = \sum_{h=0}^{m-1} r(I_{a+}, \xi, u_h g_h) + r(I_{a+}, \xi, \rho_m). \tag{2.3.3}$$

By hypothesis the last term tends asymptotically to zero uniformly in ξ ($\alpha < \xi < \beta$) for $m \to \infty$. Consequently for each fixed real q the last term in (2.3.3) is $O|\omega|^{-q}$ for sufficiently large fixed positive integer m. Applying the preceding theorem we obtain, for sufficiently large fixed positive integer m

$$\sum_{h=0}^{m-1} r(I_{a+}, Q, u_h g_h) \sim \sum_{h=0}^{m-1} u_h r(I_{a+}, Q, g_h) = \sum_{h=0}^{\infty}{}' u_h r(I_{a+}, Q, g_h) + O|\omega|^{-q}.$$

In this way we find the order relation uniform in ξ

$$r(I_{a+}, \xi, f) = \sum_{h=0}^{\infty}{}' u_h r(I_{a+}, Q, g_h) + O|\omega|^{-q}$$

valid for each fixed real q. This gives the required result.

2.4. On $\sum\limits_{n>1} n^{\sigma-1}(\log n)^{\tau} e^{-(n^{\rho}/\omega)}$. The aim of this section is to determine for large $|\omega|$ the asymptotic behavior of the sum

$$(2.4.1) \qquad \sum_{n>1} f(n) \quad \text{where} \quad f(x) = x^{\sigma-1}(\log x)^{\tau} e^{-x^{\rho}/\omega} \qquad (x>1).$$

Here the positive number ρ and the complex numbers σ and τ denote fixed numbers and we treat three cases, namely

(1) $-\dfrac{\pi}{2}+\varepsilon<\arg\omega<\dfrac{\pi}{2}-\varepsilon$, where $\varepsilon>0$ is fixed;

(2) $-\dfrac{\pi}{2}\leqq\arg\omega\leqq\dfrac{\pi}{2}$, $\operatorname{Re}\sigma<\rho<1$;

(3) $-\dfrac{\pi}{2}\leqq\arg\omega\leqq\dfrac{\pi}{2}$, $\rho=1$, $\operatorname{Re}\sigma<0$.

We shall prove

$$(2.4.2) \qquad \sum_{n=2}^{\infty} f(n) \sim \int_{3/2}^{\infty} f(x)\,dx + \sum_{h=0}^{\infty}{}' c_h\,\omega^{-h}$$

where the fixed coefficients c_h ($h=0, 1, \ldots$) can be evaluated with any desired degree of accuracy.

The convergence of the integral and the series occurring on the right-hand side of (2.4.2) is obvious in the case $-\dfrac{\pi}{2}<\arg\omega<\dfrac{\pi}{2}$. If $\arg\omega=\pm\dfrac{\pi}{2}$ and if we put $\operatorname{Im}\sigma=\lambda$, then we find for each positive ξ

$$(2.4.3) \qquad \int_{\xi}^{\infty} f(x)\,dx = -i\int_{\xi}^{\infty} \frac{x^{\operatorname{Re}\sigma-\rho}}{\rho i\omega^{-1}+\lambda x^{-\rho}}\,d\exp(i\lambda\log x-\omega^{-1}x^{\rho}).$$

The exponent of e is purely imaginary and the fraction occurring in the integrand represents a real function of x which for $\operatorname{Re}\sigma<\rho$ tends for large positive x monotonically to zero as x approaches infinity. This shows the convergence of the integral occurring in (2.4.2). In the primitive sum formula (1.2.1) of Euler

$$(2.4.4) \qquad \sum_{n>\xi} f(n) = \int_{\xi}^{\infty} f(x)\,dx - \Lambda_m(\xi+,f) + (-)^{m-1}\int_{\xi}^{\infty} f^{(m)}(x)\,\varphi_m(x)\,dx$$

the last integral converges for sufficiently large m, so that also the series occurring on the left-hand side converges.

Choose $\alpha = |\omega|^{\delta}$, where δ denotes a fixed positive number $< \rho^{-1}$. In the interval $\alpha < x < \infty$ the function $f^{(m)}(x)$ tends, uniformly in x, asymptotically to zero and

$$(2.4.5) \qquad \int_{\alpha}^{\infty} |f^{(m)}(x)|\, dx$$

tends asymptotically to zero as $m \to \infty$. This is obvious in the cases (2) and (3), so that we have only to consider the case (1). Then

$$f^{(m)}(x) = O\, x^{\sigma-1-m} (\log x)^{\tau} e^{-x^{\rho}/\omega} j(x^{\rho} \omega^{-1}),$$

where $j(u)$ is a polynomial in u of degree $\leq m$ in which the coefficients are bounded functions of x. If in this polynomial each coefficient is replaced by a constant upper bound of its absolute value, then $j(u)$ becomes a polynomial which will be denoted by $j^*(u)$. For each positive integer $m > \operatorname{Re} \sigma$ we obtain, if we put

$$\mu = 1 + (\log |\omega|)^{\operatorname{Re} \tau},$$

$$\int_{\alpha}^{\infty} |f^{(m)}(x)|\, dx$$

$$= O \mu \int_{\alpha}^{2|\omega|^{1/\rho}} x^{\operatorname{Re}\sigma-1-m}\, dx + O |\omega|^{-m/\rho} \mu \int_{2|\omega|^{1/\rho}}^{\infty} x^{\operatorname{Re}\sigma-1} j^*\left(\frac{x^{\rho}}{|\omega|}\right) e^{-\operatorname{Re}(x^{\rho}\omega^{-1})}\, dx$$

$$= O |\omega|^{-(m-\operatorname{Re}\sigma)\delta} \mu + O |\omega|^{-(m-\operatorname{Re}\sigma)/\rho} \mu \int_{2}^{\infty} y^{\operatorname{Re}\sigma-1} j^*(y) \exp(-y^{\rho} \cos \arg \omega)\, dy$$

where the last integral is bounded because in case (1) $\arg \omega$ lies between $-\frac{\pi}{2} + \varepsilon$ and $\frac{\pi}{2} - \varepsilon$. Consequently the integral (2.4.5) tends indeed asymptotically to zero for $m \to \infty$.

The function $f(x)$ and each of its derivatives tend to zero for $x \to \infty$. According to Theorem 2.2.2, applied with $a = \frac{3}{2}$ and $\beta = \infty$

$$\sum_{n=2}^{\infty} f(n) \sim \int_{3/2}^{\infty} f(x)\,dx + r\left(\frac{3}{2}, Q, f\right)$$

where Q is the periodic asymptotic neutrix with domain $\alpha < \xi < \infty$. The periodic asymptotic neutrix Q^* with domain $\alpha < \xi < 2\alpha$ has, according to the Theorems 2.1.1 and 2.3.3, the property

$$r\left(\frac{3}{2}, Q, f\right) \sim r\left(\frac{3}{2}, Q^*, f\right) \sim \sum_{h=0}^{\infty}{}' \frac{(-)^h}{h!\,\omega^h}\, r\left(\frac{3}{2}, Q^*, x^{\sigma+h\rho-1}\log^\tau x\right).$$

By the Theorems 2.1.1 and 2.2.3

$$\frac{(-)^h}{h!}\, r\left(\frac{3}{2}, Q^*, x^{\sigma+h\rho-1}\log^\tau x\right) \sim \frac{(-)^h}{h!}\, r\left(\frac{3}{2}, Q, x^{\sigma+h\rho-1}\log^\tau x\right) \sim c_h$$

where

$$c_h = \frac{(-)^h}{h!}\, r\left(\frac{3}{2}, P, x^{\sigma+h\rho-1}\log^\tau x\right).$$

This yields the required result.

The method applied in this section is very general. In a great number of problems it gives, for the difference between a given sum and the corresponding integral, an asymptotic expansion of a simple type with the property that its coefficients can be expressed by means of neutrices and can be evaluated with any desired degree of accuracy. This does not mean necessarily that we obtain in this way simple asymptotic expansions for the sum and the integral separately. For instance the integral occurring in (2.4.2) possesses a simple asymptotic expansion only if τ is an integer $\geqq 0$.

Often the coefficients occurring in the asymptotic expansions obtained in this way can be defined by means of analytic continuation of elementary functions. For instance in the special case that τ is an integer $\geqq 0$, the coefficients c_h occurring in (2.4.2) can be expressed by means of the zeta function $\zeta(s)$ of Riemann, but for this purpose we need the theory of the Hadamard neutrices which will be developed in Chapter 3.

2.5. The neutralized sum formula of Euler.

In the same way as in the Sections 2.2 and 2.3 we prove the following theorems.

Theorem 2.5.1: If $\alpha < \beta \leqq b$, where $\beta - \alpha \to \infty$ as $|\omega| \to \infty$ and if $f(x)$ $(\alpha < x < b)$ is integrable from α to an integrating neutrix I_{b-} and if $f(x)$ is asymptotically smooth in $\alpha < x < \beta$, then

$$r(I_{b-}, \xi, f) = \sum_{\xi < n < b} f(n) - \int_{\xi}^{I_{b-}} f(x)\, dx$$

is a function of ξ $(\alpha < \xi < \beta)$ which assumes a neutralized value $r(I_{b-}, Q, f)$ at the periodic asymptotic neutrix with domain $\alpha < \xi < \beta$. This residue possesses, uniformly in ξ $(\alpha < \xi < \beta)$, the asymptotic expansion

$$r(I_{b-}, Q, f) \sim r(I_{b-}, \xi, f) - \Lambda_\infty(\xi +, f).$$

If $f(x)$ is integrable from α to b, then we may write $r(b, \xi, f)$ and $r(b, Q, f)$ instead of $r(I_{b-}, \xi, f)$ and $r(I_{b-}, Q, f)$.

Theorem 2.5.2: If the conditions of the preceding theorem are satisfied and if $f(x)$ and each of its derivatives tend to zero as $x > \alpha$ tends to α, then

$$r(I_{b-}, Q, f) \sim \lim_{\xi \to \alpha} r(I_{b-}, \xi, f)$$

provided that the limit occurring on the right-hand side exists.

Theorem 2.5.3: Assume $\beta \leqq b$. Assume that $f(x)$ is integrable from $-\infty +$ to an integrating neutrix I_{b-}. If $f(x)$ is asymptotically smooth in the interval $-\infty < x < \beta$ and if, for each fixed sufficiently large positive integer m, $f^{(m)}(x)$ tends to zero as $x \to -\infty$, then $r(I_{b-}, Q, f)$ and $r(I_{b-}, -P, f)$ exist and

$$r(I_{b-}, Q, f) \sim r(I_{b-}, -P, f).$$

Here Q is the periodic asymptotic neutrix with domain $-\infty < \xi < \beta$.

Theorem 2.5.4: If $a - \beta$ is a positive number tending to infinity as $|\omega| \to \infty$ and if $f(x)$ is asymptotically smooth in the interval $\beta < x \leqq a$, then

$$r(a, Q, f) \sim \Lambda_\infty (a-, f).$$

where Q is the periodic asymptotic neutrix with domain $\beta < \xi < a$.

Theorem 2.5.5: Assume $\alpha < \beta \leqq b$, where $\beta - \alpha \to \infty$ as $|\omega| \to \infty$, Let Q be the periodic asymptotic neutrix with domain $\alpha < \xi < \beta$. Let u_h ($h = 0, 1, \ldots$) be a number independent of x which is asymptotically finite for each fixed integer $h \geqq 0$. If the functions $g_h(x)$ ($h = 0, 1, \ldots$) defined for $\alpha < x < b$ are integrable from $\alpha+$ to an integrating neutrix I^{b-}, if $r(I_{b-}, Q, g_h)$ exists and $u_h r(I_{b-}, Q, g_h)$ tends asymptotically to zero as $h \to \infty$, then each function $f(x)$ integrable from $\alpha+$ to I_{b-} such that

$$r\left(I_{b-}, \xi, f - \sum_{h=0}^{m-1} u_h g_h\right)$$

tends asymptotically to zero in the interval $\alpha < \xi < b$, uniformly in ξ, as $m \to \infty$, has the property that $r(I_{b-}, Q, f)$ exists and possesses the asymptotic expansion

$$r(I_{b-}, Q, f) \sim \sum_{h=0}^{\infty}{}' u_h r(I_{b-}, Q, g_h).$$

The following theorem, the main theorem of this paper, can be summarized as follows; under a general condition a sum $\sum\limits_{a<n<b}$ is asymptotically equal to the corresponding integral from I_{a+} to I_{b-} plus the sum of the residues at I_{a+} and I_{b-}.

Theorem 2.5.6: Neutralized Sum Formula of Euler: Assume

$$a \leqq \alpha < \beta \leqq \alpha^* < \beta^* \leqq b$$

where $\beta - \alpha$ and $\beta^* - \alpha^*$ tend to infinity as $|\omega| \to \infty$. Let Q and Q^* be the periodic asymptotic neutrices respectively with domains $\alpha < x < \beta$ and $\alpha^* < x < \beta^*$. If $f(x)$ is integrable from an integrating neutrix I_{a+} to an integrating neutrix I_{b-} and if $f(x)$ is asymptotically smooth in the interval

$\alpha < x < \beta^*$, then

$$(2.5.1) \qquad \sum_{a<n<b} f(n) \sim \int_{I_{a+}}^{I_{b-}} f(x)\,dx + r(I_{a+}, Q, f) + r(I_{b-}, Q^*, f).$$

Proof: If $\alpha < \xi < \beta$ and $\alpha^* < \xi^* < \beta^*$, then we have according to the primitive sum formula (1.2.1) of Euler

$$\sum_{\xi \leqq n \leqq \xi^*} f(n) - \int_{\xi}^{\xi^*} f(x)\,dx$$

$$= \Lambda_m(\xi^*_+, f) - \Lambda_m(\xi_-, f) + (-)^{m-1} \int_{\xi}^{\frac{1}{2}(\alpha+\beta^*)} f^{(m)}(x)\,\varphi_m(x)\,dx$$

$$+ (-)^{m-1} \int_{\frac{1}{2}(\alpha+\beta^*)}^{\xi^*} f^{(m)}(x)\,\varphi_m(x)\,dx$$

so that

$$(2.5.2) \qquad \sum_{a<n<b} f(n) - \int_{I_{a+}}^{I_{b-}} f(x)\,dx = u(\xi) + v(\xi^*)$$

where

$$(2.5.3) \qquad u(\xi) = r(I_{a+}, \xi, f) - \Lambda_m(\xi -, f) + (-)^{m-1} \int_{\xi}^{\frac{1}{2}(\alpha+\beta^*)} f^{(m)}(x)\,\varphi_m(x)\,dx$$

and

$$v(\xi^*) = r(I_{b-}, \xi^*, f) + \Lambda_m(\xi^*_+, f) + (-)^{m-1} \int_{\frac{1}{2}(\alpha+\beta^*)}^{\xi^*} f^{(m)}(x)\,\varphi_m(x)\,dx.$$

For each fixed real q the last term occurring in (2.5.3) is $O\,|\omega|^{-q}$, if the fixed positive integer m is large enough. Since each term of the sum $\Lambda_m(\xi -, f)$ is negligible in Q, we find therefore for each fixed real q

$$u(Q) = r(I_{a+}, Q, f) + O\,|\omega|^{-q}$$

hence

$$u(Q) \sim r(I_{a+}, Q, f).$$

In the same way we find

$$v(Q^*) \sim r(I_{b-}, Q^*, f).$$

This gives the required result.

CHAPTER 3. HADAMARD NEUTRICES

3.1. Definition of Hadamard neutrices.

In this chapter I restrict myself to a special kind of Hadamard neutrices, whereas a more general definition will be given in Section 7.2.

Definition: If a is real, then a Hadamard neutrix H_{a+} is a neutrix with domain $a<\xi<\beta$, where $\beta>a$ (β may be ∞), formed by functions of the form

(3.1.1) $$\nu(\xi) = \lambda(\xi) + o(1);$$

here $o(1)$ denotes a function of ξ ($a < \xi < \beta$) which tends to zero as $\xi \to a$; $\lambda(\xi)$ is a linear combination, with constant coefficients, of functions of the form

(3.1.2) $$(\xi - a)^\sigma \log^k (\xi - a)$$

where the exponents σ are arbitrary complex numbers and where the exponents k are arbitrary integers $\geqq 0$, with the restriction that the couple $\sigma = 0$, $k=0$ is not admitted.

If a is real, then a Hadamard neutrix H_{a-} is a neutrix with domain $\beta<\xi<a$, where $\beta<a$ (β may be $-\infty$), formed by functions of the form (3.1.1); here $o(1)$ denotes a function of ξ ($\beta<\xi<a$) which tends to zero as $\xi \to a$; $\lambda(\xi)$ is a linear combination, with constant coefficients, of functions of the form

(3.1.3) $$(a - \xi)^\sigma \log^k (a - \xi)$$

where the exponents σ are arbitrary complex numbers and where the exponents k are arbitrary integers $\geqq 0$, with the restriction that the couple $\sigma=0$, $k=0$ is not admitted.

If a is complex, then $H_\infty(a)$ is a neutrix with domain $\beta < \xi < \infty$, where β is real and $a-\beta$ is not a positive number, formed by functions

of the form (3.1.1); here $o(1)$ denotes a function of ξ $(\beta<\xi<\infty)$ which tends to zero as $\xi\to\infty$; $\lambda(\xi)$ is a linear combination, with constant coefficients, of functions of the form (3.1.2), where the exponents σ are arbitrary complex numbers and the exponents k are arbitrary integers $\geqq 0$, with the restriction that the couple $\sigma=0$, $k=0$ is not admitted.

If a is complex, then $H_{-\infty}(a)$ is a neutrix with domain $-\infty<\xi<\beta$, where β is real and $\beta-a$ is not a positive number, formed by functions of the form (3.1.1): here $o(1)$ denotes a function of ξ $(-\infty<\xi<\beta)$ which tends to zero as $\xi\to-\infty$; $\lambda(\xi)$ is a linear combination, with constant coefficients, of functions of the form (3.1.3), where the exponents σ are arbitrary complex numbers and the exponents k are integers $\geqq 0$, with the restriction that the couple $\sigma=0$, $k=0$ is not admitted.

In Section 7.2 we show that in each of these four cases the neutrix condition is satisfied.

That the Hadamard neutrices are integrating neutrices follows from Theorem 1.4.1.

Theorem 3.1.1: Consider an integral of the form

$$j=\int_H^b (x-a)^{\sigma-1}\log^k(x-a)\,dx \tag{3.1.4}$$

where $b>a$, σ is complex, k is an integer $\geqq 0$, $H=H_{a+}$ or $H_\infty(a)$. Then j is equal to

$$\psi(b-a,\sigma,k)=\begin{cases}(k+1)^{-1}\log^{k+1}(b-a) & \text{if } \sigma=0\\ k!\,(b-a)^\sigma\sum_{h=0}^{k}(-)^h\dfrac{\log^{k-h}(b-a)}{(k-h)!\,\sigma^{h+1}}=\left(\dfrac{\partial}{\partial\sigma}\right)^k\dfrac{(b-a)^\sigma}{\sigma} & \text{if } \sigma\neq 0\end{cases} \tag{3.1.5}$$

If $\sigma\neq 0$, then

$$j=(-)^k k!\,\sigma^{-k-1}+\int_H^b (x-a)^{-1}\log^k(x-a)\,dx+o(1) \tag{3.1.6}$$

where $o(1)$ denotes a function of σ which is analytic and equal to zero at $\sigma=0$.

Proof: We have in the interval $a<\xi<b$

$$\int_{\xi}^{b} (x-a)^{-1} \log^k (x-a)\, dx = (k+1)^{-1} \log^{k+1} (b-a) - (k+1)^{-1} \log^{k+1} (\xi-a)$$

where the last term is negligible in H. This gives (3.1.5) for $\sigma = 0$. If $\sigma \neq 0$, then the function ψ defined in (3.1.5) has the property

$$\frac{\partial}{\partial x} \psi (x-a, \sigma, k) = (x-a)^{\sigma-1} \log^k (x-a)$$

so that for $a<\xi<b$

$$\int_{\xi}^{b} (x-a)^{\sigma-1} \log^k (x-a)\, dx = \psi (b-a, \sigma, k) - \psi (\xi-a, \sigma, k)$$

where the last term is negligible in H. This gives (3.1.5) for $\sigma \neq 0$, since according to the rule of Leibniz

$$\left(\frac{\partial}{\partial\sigma}\right)^k \sigma^{-1} (b-a)^\sigma = \sum_{h=0}^{k} \binom{k}{h} (-)^h\, h!\, \sigma^{-h-1} (b-a)^\sigma \log^{k-h} (b-a).$$

Finally

$$\begin{aligned} j &= \left(\frac{\partial}{\partial\sigma}\right)^k \sigma^{-1} e^{\sigma \log (b-a)} = \left(\frac{\partial}{\partial\sigma}\right)^k \sum_{h=0}^{\infty} \frac{\sigma^{h-1}}{h!} \log^h (b-a) \\ &= (-)^k k!\, \sigma^{-k-1} + (k+1)^{-1} \log^{k+1} (b-a) + o(1). \end{aligned}$$

This completes the proof.

In the same way we prove

Theorem 3.1.2: Consider an integral of the form

$$j = \int_{b}^{H} (a-x)^{\sigma-1} \log^k (a-x)\, dx$$

where $b<a$, σ is complex, k is an integer $\geqq 0$, $H = H_{a-}$ or $H_{-\infty}(a)$. Then j is equal to $\psi(a-b, \sigma, k)$, where ψ is defined by (3.1.5). If $\sigma \neq 0$, then

$$j = (-)^k k!\, \sigma^{-k-1} + \int_{b}^{H} (a-x)^{-1} \log^k (a-x)\, dx + o(1)$$

where $o(1)$ denotes a function of σ which is analytic and equal to zero at $\sigma = 0$.

Theorem 3.1.3: If $\omega > 0$, $a < b$, $\alpha = \omega^{-1} a$, $\beta = \omega^{-1} b$, $p \neq 0$, and k is an integer ≥ 0, then we have for each $\sigma \neq 0$

$$(3.1.7) \qquad \int_{H_{a+}}^{b} (x-a)^{\sigma-1} (\log p\,(x-a))^k\,dx = \omega^\sigma \int_{H_{\alpha+}}^{\beta} (y-\alpha)^{\sigma-1} (\log p\omega\,(y-\alpha))^k\,dy$$

but

$$(3.1.8) \qquad \int_{H_{a+}}^{b} (x-a)^{-1} (\log p\,(x-a))^k\,dx$$

$$= \frac{(\log p\omega)^{k+1} - (\log p)^{k+1}}{k+1} + \int_{H_{\alpha+}}^{\beta} (y-\alpha)^{-1} (\log p\omega\,(y-\alpha))^k\,dy\,.$$

Remark: In Section 7.6 I shall develop the theory of a new integration variable. The first term occurring on the right-hand side of (3.1.8) warns us that even the simple substitution $x - a = \omega\,(y-\alpha)$ may lead to a supplementary term.

Proof: (1) Let us first treat the special case $p = 1$. If $p = 1$ and $\sigma \neq 0$, then the left-hand side of (3.1.7) is according to (3.1.5) equal to

$$\left(\frac{\partial}{\partial\sigma}\right)^k \frac{(b-a)^\sigma}{\sigma} = \left(\frac{\partial}{\partial\sigma}\right)^k \frac{\omega^\sigma (\beta-\alpha)^\sigma}{\sigma} = \sum_{h=0}^{k} \binom{k}{h} \left\{\left(\frac{\partial}{\partial\sigma}\right)^{k-h} \omega^\sigma\right\} \left(\frac{\partial}{\partial\sigma}\right)^h \frac{(\beta-\alpha)^\sigma}{\sigma}$$

$$= \omega^\sigma \sum_{h=0}^{k} \binom{k}{h} (\log \omega)^{k-h} \int_{H_{\alpha+}}^{\beta} (y-\alpha)^{\sigma-1} \log^h (y-\alpha)\,dy$$

$$= \omega^\sigma \int_{H_{\alpha+}}^{\beta} (y-\alpha)^{\sigma-1} \log^k (\omega y - \omega\alpha)\,dy\,.$$

Furthermore, if $p=1$, then the left-hand side of (3.1.8) is equal to $(k+1)^{-1} \log^{k+1} (b-a) = (k+1)^{-1} (\log (\beta-\alpha) + \log \omega)^{k+1}$

$$= (k+1)^{-1} \sum_{h=0}^{k+1} \binom{k+1}{h} \log^h \omega \log^{k+1-h} (\beta-\alpha)\,.$$

The right-hand side of (3.1.8) is equal to

$$(k+1)^{-1}\log^{k+1}\omega+\sum_{h=0}^{k}\binom{k}{h}\log^h\omega\int_{H_{\alpha+}}^{\beta}(y-\alpha)^{-1}\log^{k-h}(y-\alpha)\,dy$$

$$=(k+1)^{-1}\log^{k+1}\omega+\sum_{h=0}^{k}\binom{k}{h}\log^h\omega\,\frac{\log^{k+1-h}(\beta-\alpha)}{k+1-h}$$

and has therefore the same value.

(2) Now we treat the general case. The left hand side of (3.1.7) is equal to

$$\sum_{h=0}^{k}\binom{k}{h}(\log p)^{k-h}\int_{H_{a+}}^{b}(x-a)^{\sigma-1}\log^h(x-a)\,dx$$

and is therefore, according to (3.1.7), applied with $p=1$, $\sigma\neq 0$ equal to

$$\omega^\sigma\sum_{h=0}^{k}\binom{k}{h}(\log p)^{k-h}\int_{H_{\alpha+}}^{\beta}(y-\alpha)^{\sigma-1}\log^h(\omega y-\omega\alpha)\,dy$$

which is equal to the right-hand side of (3.1.7) Finally the left-hand side of (3.1.8) has the value

$$\sum_{h=0}^{k}\binom{k}{h}(\log p)^{k-h}\int_{H_{a+}}^{b}(x-a)^{-1}\log^h(x-a)\,dx$$

and is therefore, according to (3.1.8), applied with $p=1$, equal to

$$\sum_{h=0}^{k}\binom{k}{h}(\log p)^{k-h}\left\{\frac{\log^{h+1}\omega}{h+1}+\int_{H_{\alpha+}}^{\beta}(y-\alpha)^{-1}(\log\omega(y-\alpha))^h\,dy\right\}$$

$$=\frac{(\log p\omega)^{k+1}-(\log p)^{k+1}}{k+1}+\int_{H_{\alpha+}}^{\beta}(y-\alpha)^{-1}(\log p\omega(y-\alpha))^k\,dy\,.$$

3.2. Hadamard expansions.

Definition: (1) Let a be real. A function $f(x)$ is said to possess at $a+$ a Hadamard expansion if it possesses at the points $x>a$ in the neighborhood of a an asymptotic expansion of the form

$$(3.2.1) \qquad f(x) \sim \sum_{h=0}^{\infty}{}' c_h (x-a)^{\sigma_h} p_h(\log(x-a))$$

where the exponents σ_h are arbitrary complex numbers with $\operatorname{Re}\sigma_h \to \infty$ as $h \to \infty$ and where $p_h(t)$ is a polynomial in t.

Formula (3.2.1) means that $f(x)$ can be written, for each fixed real q, in the form

$$(3.2.2) \qquad f(x) = \sum_{\substack{h \\ \operatorname{Re}\sigma_h \leqq q}} c_h (x-a)^{\sigma_h} p_h(\log(x-a)) + O(x-a)^q .$$

The numbers $\sigma_0, \sigma_1, \ldots$ are called the exponents in the Hadamard expansion.

(2) Let a be complex. A function $f(x)$ is said to possess at ∞ a Hadamard expansion with parameter a if it possesses for large positive x an asymptotic expansion of the form (3.2.1), where the exponents σ_h are arbitrary complex numbers with $\operatorname{Re}\sigma_h \to -\infty$ as $h \to \infty$ and where $p_h(t)$ is a polynomial in t.

(3) Let a be real. A function $f(x)$ is said to possess at $a-$ an Hadamard expansion if it possesses at the points $x<a$ in the neighborhood of a an asymptotic expansion of the form

$$(3.2.3) \qquad f(x) = \sum_{h=0}^{\infty}{}' c_h (a-x)^{\sigma_h} p_h(\log(a-x))$$

where the exponents σ_h are arbitrary complex numbers with $\operatorname{Re}\sigma_h \to \infty$ as $h \to \infty$ and where $p_h(t)$ is a polynomial in t.

(4) Let a be complex. A function $f(x)$ is said to possess at $-\infty$ a Hadamard expansion with parameter a if it possesses for negative x with large $|x|$ an asymptotic expansion of the form (3.2.3), where the exponents σ_h are arbitrary complex numbers with $\operatorname{Re}\sigma_h \to -\infty$ as $h \to \infty$ and where $p_h(t)$ is a polynomial in t.

Theorem 3.2.1: (1) If $-\infty < a < b$, then each function which is integrable from $a+$ to b and possesses at $a+$ a Hadamard expansion is integrable from H_{a+} to b.

(2) If a is complex and $-\infty \leqq b < \infty$, then each function which is integrable from b to $\infty-$ and possesses at ∞ a Hadamard expansion with parameter a is integrable from b to $H_\infty(a)$.

(3) If $b<a<\infty$, then each function which is integrable from b to $a-$ and possesses at $a-$ a Hadamard expansion is integrable from b to H_{a-}.

(4) If a is complex and $-\infty<b\leqq\infty$, then each function which is integrable from $-\infty+$ to b and possesses at $-\infty$ a Hadamard expansion with parameter a is integrable from $H_{-\infty}(a)$ to b.

Proof: We give the proof for the case (1). The proof in the cases (2), (3) and (4) runs in the same way. Applying (3.2.2) with $q=0$ we find

$$f(x) = \sum_{\substack{h \\ \operatorname{Re}\sigma_h \leqq 0}} c_h (x-a)^{\sigma_h} p_h(\log(x-a)) + \rho(x)$$

where $\rho(x)$ is bounded for $x>a$ in the neighborhood of a. This bounded function is integrable from $a+$ to b, therefore from a to b. Each term occurring in the sum $\sum\limits_h$ is integrable from H_{a+} to b according to Theorem 3.1.1. This gives the required result.

3.3. Normal Hadamard expansions.

Definition: The Hadamard expansions (3.2.1) and (3.2.3) in the neighborhood of a point a are called normal if $\sigma_0=0$; $p_0(t)$ is a number $\neq 0$ independent of t and finally $\operatorname{Re}\sigma_h>0$ for $h=1, 2, \ldots$. The Hadamard expansions (3.2.1) and (3.2.3) in the neighborhood of ∞ or $-\infty$ are called normal if $\sigma_0=0$; $p_0(t)$ is a number $\neq 0$ independent of t and finally $\operatorname{Re}\sigma_h<0$ for $h=1, 2, \ldots$. In a normal Hadamard expansion we always assume that the exponents $\sigma_0, \sigma_1, \ldots$ form a semigroup; this means that the sum of any two exponents is again an exponent. This does not mean a restriction since we may always add terms with coefficients zero.

The two following theorems are obvious.

Theorem 3.3.1 (Product theorem): The product of two functions which possess at $a+$ Hadamard expansions respectively with exponents σ_h ($h=0, 1, \ldots$) and τ_k ($k=0, 1, \ldots$), possesses at $a+$ a Hadamard expansion with exponents $\sigma_h+\tau_k$ ($h\geqq 0$; $k\geqq 0$).

This is also true with $a+$ replaced by $a-$. It is even true with a replaced by $\pm\infty$ if the two given Hadamard expansions have the same parameter a.

Theorem 3.3.2 (Power theorem): If $g(x)$ possesses at $a+$ a normal Hadamard expansion, then $g^{-s}(x)$ possesses for each complex s at a_+ a normal Hadamard expansion with the same exponents. Each coefficient in this new expansion is u^{-s} times a polynomial in s, where u denotes the constant term occurring in the normal Hadamard expansion of $g(x)$; by hypothesis this constant term u is $\neq 0$.

This statement remains true with $a+$ replaced by $a-$ or ∞ or $-\infty$.

3.4. Analytic continuation for Hadamard integrals.

Theorem 3.4.1: Assume that $-\infty < a < b < \infty$, that $f(x)$ and $g(x) \neq 0$ are continuous in the interval $a < x \leqq b$, that $f(x)$ possesses at $a+$ a Hadamard expansion with exponents $\sigma_h - 1$ $(h = 0, 1, \ldots)$ and that $(x-a)^{-1} g(x)$ possesses at $a+$ a normal Hadamard expansion with exponents $\tau_0, \tau_1, \ldots$.

Then

$$\chi(s) = \int_{H_{\alpha+}}^{b} g^{-s}(x) f(x)\, dx$$

represents an analytic function of s everywhere in the complex s-plane, the points $\sigma_h + \tau_k$ $(h \geqq 0\ ;\ k \geqq 0)$ excepted. Each point $\sigma_h + \tau_k$ is a pole of the function $\chi(s)$.

By choosing $g(x) = x - a$ we obtain in particular that the function

$$\chi_1(s) = \int_{H_{a+}}^{b} (x-a)^{-s} f(x)\, dx$$

represents an analytic function of s everywhere in the complex s-plane, the points $\sigma_0, \sigma_1, \ldots$ excepted. Each point σ occurring in the sequence $\sigma_0, \sigma_1, \ldots$ is a pole of $\chi_1(s)$; $\chi_1(\sigma)$

is the constant term in the Laurent expansion of the function $\chi_1(s)$ according to powers of $s-\sigma$. More precisely, at the points $s \neq \sigma$ in the neighborhood of σ we have

$$\chi_1(s) = -\sum_{\substack{h \\ \sigma_h \quad \sigma}} k_h!\, c_h (s-\sigma)^{-k_h-1} + \chi_1(\sigma) + o(1) \tag{3.4.1}$$

where $o(1)$ denotes a function of s which is analytic and equal to zero at $s=\sigma$; the numbers c_h and k_h are the numbers occurring in the Hadamard expansion

$$f(x) \sim \sum_{h=0}^{\infty}{}' c_h (x-a)^{\sigma_h-1} \log^{k_h}(x-a) \tag{3.4.2}$$

valid at $a+$ by hypothesis.

Remark: In the more general case of the function $\chi(s)$ we shall find in Section 7.5 a similar, but more complicated result for the value of $\chi(\sigma)$, where σ is a number occuring in the sequence $\sigma_h + \tau_k$ $(h \geqq 0 \ ; \ k \geqq 0)$.

Proof: Using the Theorems 3.3.1 and 3.3.2 we find at the points $x > a$ in the neighborhood of a for $g^{-s}(x)\, f(x)$ a Hadamard expansion of the form

$$u^{-s} \sum_{h \geqq 0\,,\, k \geqq 0}{}' c_{hk}(s)\, (x-a)^{\sigma_h+\tau_k-s-1}\, p_{hk}(\log(x-a))$$

where $c_{hk}(s)$ is a polynomial in s, where $p_{hk}(t)$ is a polynomial in t and where u is the constant term in the normal Hadamard expansion of $(x-a)^{-1}\, g(x)$; this constant term u is $\neq 0$ by hypothesis.

If s lies in a bounded region Δ, then we can write

$$g^{-s}(x)\, f(x) = u^{-s} \sum\nolimits_1 c_{hk}(s)\, (x-a)^{\sigma_h+\tau_k-s-1}\, p_{hk}(\log(x-a)) + \rho(x\,,s)$$

where Σ_1 is a finite sum extended over integers $h \geqq 0$ and $k \geqq 0$ and where

$$\int_a^b \rho(x\,,\, s)\, dx$$

denotes a function of s which is analytic in Δ. Consequently

$$\int_{H_{a+}}^{b} g^{-s}(x)\, f(x)\, dx$$

$$= u^{-s} \sum\nolimits_1 c_{hk} \int_{H_{a+}}^{b} (x-a)^{\sigma_h+\tau_k-s-1}\, p_{hk}(\log(x-a))\, dx + \int_a^b \rho(x, s)\, dx$$

is according to Theorem 3.1.1 analytic in Δ, except at the points $\sigma_h + \tau_k$ ($h \geqq 0$, $k \geqq 0$); these points are poles. This gives the required result for $\chi(s)$.

Let σ be a number occurring in the sequence $\sigma_0, \sigma_1, \ldots$. Using (3.4.2) we obtain

$$(x-a)^{-s} f(x) = \sum\nolimits_2 c_h (x-a)^{\sigma_h-s-1} \log^{k_h}(x-a) + \rho_1(x, s)$$

where Σ_2 is a finite sum extended over integers $h \geqq 0$ and $k \geqq 0$ and where the integral

$$\int_a^b \rho_1(x, s)\, dx$$

represents a function of s which is analytic at σ. In this way we find

$$\chi_1(s) = \sum\nolimits_2 c_h \int_{H_{a+}}^{b} (x-a)^{\sigma_h-s-1} \log^{k_h}(x-a)\, dx + \int_a^b \rho_1(x, s)\, dx\,.$$

According to Theorem 3.1.1 each term on the right-hand side represents an analytic function of s at σ, except the terms with $\sigma_h = \sigma$ and by means of (3.1.6) we write the right-hand side in the form

$$\sum_{\substack{h \\ \sigma_h=\sigma}} (-)^{k_h} k_h!\, c_h(\sigma-s)^{-k_h-1} + \sum\nolimits_2 c_h \int_{H_{a+}}^{b} (x-a)^{\sigma_h-\sigma-1} \log^{k_h}(x-a)\, dx$$

$$+ \int_a^b \rho_1(x, \sigma)\, dx + o(1)$$

where $o(1)$ denotes a function of s which is analytic and equal to zero at $s=\sigma$. This gives the required result for $\chi_1(s)$.

In the same way we prove the following three theorems.

Theorem 3.4.2: Assume that $-\infty < b < a < \infty$, that $f(x)$ and $g(x) \neq 0$ are continuous in the interval $b \leqq x < a$, that $f(x)$ possesses at $a-$ a Hadamard expansion with exponents $\sigma_h - 1$ $(h = 0, 1, \ldots)$ and that $(a-x)^{-1} g(x)$ possesses at $a-$ a normal expansion with exponents $\tau_0, \tau_1, \ldots$.

Then

$$\chi(s) = \int_b^{H_{a-}} g^{-s}(x) f(x)\, dx$$

represents an analytic function of s everywhere in the complex s-plane, the points $\sigma_h + \tau_k$ $(h \geqq 0\,;\ k \geqq 0)$ excepted. Each point $\sigma_h + \tau_k$ is a pole of the function $\chi(s)$.

By choosing $g(x) = a - x$ we obtain in particular that

$$\chi_1(s) = \int_b^{H_{a-}} (a-x)^{-s} f(x)\, dx$$

represents an analytic function of s everywhere in the complex s-plane, the points $\sigma_0, \sigma_1, \ldots$ excepted. Each point σ occurring in the sequence $\sigma_0, \sigma_1, \ldots$ is a pole of $\chi_1(s)$; $\chi_1(\sigma)$ is the constant term in the Laurent expansion of $\chi_1(s)$ according to powers of $s - \sigma$ and we have at the points $s \neq \sigma$ in the neighborhood of σ

$$\chi_1(s) = -\sum_{\substack{h \\ \sigma_h = \sigma}} k_h!\, c_h (s-\sigma)^{-k_h-1} + \chi_1(\sigma) + o(1) \tag{3.4.3}$$

where $o(1)$ denotes a function of s which is analytic and equal to zero at $s = \sigma$; the numbers c_h and k_h are the numbers occurring in the Hadamard expansion

$$f(x) \sim \sum_{h=0}^{\infty}{}' c_h (a-x)^{\sigma_h - 1} \log^{k_h}(a-x)\,. \tag{3.4.4}$$

Theorem 3.4.3: Let a be a complex number. Assume that $-\infty < b < \infty$, that $f(x)$ and $g(x) \neq 0$ are continuous in the interval $b \leqq x < \infty$, that $f(x)$ possesses at ∞ a Hadamard

expansion with parameter a and exponents $\sigma_h - 1$ ($h=0, 1, \ldots$) and that $(x-a)^{-1} g(x)$ possesses at ∞ a normal Hadamard expansion with parameter a and exponents $\tau_0, \tau_1, \ldots$. Then

$$\chi(s) = \int_{H_\infty(a)}^{b} g^{-s}(x) f(x)\, dx$$

represents an analytic function of s everywhere in the complex s-plane, the points $\sigma_h + \tau_k$ ($h \geqq 0$; $k \geqq 0$) excepted. Each point $\sigma_h + \tau_k$ is a pole of $\chi(s)$.

By choosing $g(x) = x - a$ we obtain in particular that

$$\chi_1(s) = \int_{H_\infty(a)}^{b} (x-a)^{-s} f(x)\, dx$$

represents an analytic function of s everywhere in the complex s-plane, the points σ_0, σ_1, ... excepted. Each point σ occurring in the sequence σ_0, σ_1, ... is a pole of $\chi_1(s)$; $\chi_1(\sigma)$ is the constant term in the Laurent expansion of $\chi_1(s)$ according to powers of $s - \sigma$ and (3.4.3) holds at the points $s \neq \sigma$ in the neighborhood of σ.

Theorem 3.4.4: Let a be a complex number. Assume that $-\infty < b < \infty$, that $f(x)$ and $g(x)$ are continuous in the interval $-\infty < x \leqq b$, that $f(x)$ possesses at $-\infty$ a Hadamard expansion with parameter a and exponents $\sigma_h - 1$ ($h=0, 1, \ldots$) and that $(x-a)^{-1} g(x)$ possesses at $-\infty$ a normal Hadamard expansion with parameter a and exponents $\tau_0, \tau_1, \ldots$. Then

$$\chi(s) = \int_{b}^{H_{-\infty}(a)} g^{-s}(x) f(x)\, dx$$

respresents an analytic function of s everywhere in the complex s-plane, the points $\sigma_h + \tau_k$ ($h \geqq 0$, $k \geqq 0$) excepted. Each point $\sigma_h + \tau_k$ is a pole of $\chi(s)$.

By choosing $g(x) = a - x$ we obtain in particular that

$$\chi_1(s) = \int_{b}^{H_{-\infty}(a)} (a-x)^{-s} f(x)\, dx$$

represents an analytic function of s everywhere in the complex s-plane, the points σ_0, σ_1, ... excepted. Each point σ occurring in the sequence σ_0, σ_1, ... is a pole of $\chi_1(s)$; $\chi_1(\sigma)$ is the constant term in the Laurent expansion of $\chi_1(s)$ according to powers of $s-\sigma$ and (3.4.1) holds at the points $s \neq \sigma$ in the neighborhood of σ.

3.5. On functions of the form $g^{-s}(x) f(x)$.

Theorem 3.5.1: If two functions $u(x)$ and $v(x)$, defined for sufficiently large positive x, are infinitely often differentiable in such a way that, for large positive x, $u^{(m)}(x)$ and $v^{(m)}(x)$ tend asymptotically to zero as $m \to \infty$, then, for large positive x, $(d/dx)^n\, uv$ tends asymptotically to zero as $n \to \infty$.

Proof: It follows from the conditions imposed on $u(x)$ and $v(x)$ that there exists a constant integer $p \geqq 0$ such that for large positive x and for each constant integer $n \geqq p$

$$u^{(n)}(x) = O(1) \qquad \text{and} \qquad v^{(n)}(x) = O(1)$$

hence

$$u^{(n)}(x) = O\, x^{p-n} \qquad \text{and} \qquad v^{(n)}(x) = O\, x^{p-n} \qquad (0 \leqq n \leqq p).$$

In this way we find for each constant integer $n \geqq 0$

$$u^{(n)}(x) = O\, x^{p} \qquad \text{and} \qquad v^{(n)}(x) = O\, x^{p}. \tag{3.5.1}$$

Let q be an arbitrary real constant. There exists an integer $n_q \geqq 0$ such that for each fixed integer $h \geqq \frac{1}{2} n_q$

$$u^{(h)}(x) = O\, x^{-p-q} \qquad \text{and} \qquad v^{(h)}(x) = O\, x^{-p-q}. \tag{3.5.2}$$

If $n \geqq n_q$, then each term in the sum

$$\left(\frac{d}{dx}\right)^n uv = \sum_{h=0}^{n} \binom{n}{h} u^{(h)}(x)\, v^{(n-h)}(x)$$

is according to (3.5.1) and (3.5.2) at most

$$O\, x^{p} \cdot O\, x^{-p-q} = O\, x^{-q}.$$

This gives the required result.

Theorem 3.5.2: If for sufficiently large positive x the function $g(x)$ is $\neq 0$ and infinitely often differentiable in

such a way that for each constant positive integer h

$$g^{(h)}(x) = O\, x^{-h\delta} g(x) \tag{3.5.3}$$

where $\delta > 0$ is independent of x and h, and if s lies in a bounded region Δ of the complex plane, then the order relation

$$\left(\frac{d}{dx}\right)^h g^{-s}(x) = O\, x^{-h\delta}\, g^{-s}(x) \tag{3.5.4}$$

holds in Δ, uniformly in s, for large positive x and for each constant integer $h \geqq 0$.

Proof: The left-hand side of (3.5.4) can be written as a linear combination of functions of the form

$$g^{-s-l}(g')^{k_1}(g'')^{k_2} \ldots (g^{(h)})^{k_h} \tag{3.5.5}$$

where $k_1, \ldots, k_h$ are integers $\geqq 0$ with $k_1 + 2k_2 + \ldots + hk_h = h$ and $k_1 + k_2 + \ldots + k_h = l$. The coefficients in this linear combination are bounded, since they can be written as polynomials in s with constant coefficients. We can write (3.5.5) in the form

$$\begin{aligned} g^{-s}\left(\frac{g'}{g}\right)^{k_1}\left(\frac{g''}{g}\right)^{k_2} \cdots \left(\frac{g^{(h)}}{g}\right)^{k_h} &= O\, x^{-(k_1+2k_2+\ldots+hk_h)\delta}\, g^{-s}(x) \\ &= O\, x^{-h\delta}\, g^{-s}(x) \end{aligned}$$

which gives the required result.

Theorem 3.5.3: Assume for large positive x that $f(x)$ is infinitely often differentiable and that $f^{(m)}(x)$ tends asymptotically to zero as $m \to \infty$. Assume that $g(x)$ satisfies the condition of the preceding theorem, that for large positive x, $g(x)$ and $g^{-1}(x)$ are asymptotically finite and that $\arg g(x)$ is bounded. Then the function

$$f_s(x) = g^{-s}(x)\, f(x)$$

where s lies in a bounded region, has the property that, for large positive x, $f_s^{(m)}(x)$ tends asymptotically to zero, uniformly in s, as $m \to \infty$.

Proof: According to the preceding theorem

$$\left(\frac{d}{dx}\right)^h g^{-s}(x)$$

tends for large positive x asymptotically to zero, uniformly in s, as $h \to \infty$, so that the assertion follows from Theorem 3.5.1.

3.6. Asymptotic behavior of sums with fixed terms.

Theorem 3.6.1: Assume that, for large positive x, the fixed function $f(x)$ is infinitely often differentiable, $f^{(m)}(x)$ tends asymptotically to zero for $m \to \infty$ and $f(x)$ possesses at infinity a Hadamard expansion with parameter a and exponents $\sigma_h - 1$ $(h = 0, 1, \ldots)$; here a denotes a fixed number. If the numbers $f(n)$ $(n = 1, 2, \ldots)$ are defined, then the function $f_s(x) = (x-a)^{-s} f(x)$ has the property that for large positive ω

$$\sum_{0<n<\omega} f_s(n) \sim -\int_{\omega}^{H_\infty(a)} f_s(x)\,dx + \chi(s) + \Lambda_\infty(\omega-, f_s) \tag{3.6.1}$$

where in the whole complex s-plane, the points $\sigma_0, \sigma_1, \ldots$ excepted, $\chi(s)$ is the analytic continuation of the function represented for sufficiently large $\operatorname{Re} s$ by the series $\sum_{n>0} f_s(n)$. Each point σ_h $(h = 0, 1, \ldots)$ is a pole of $\chi(s)$ and $\chi(\sigma_h)$ is the constant term in the Laurent expansion of $\chi(s)$ according to powers of $s - \sigma_h$. Finally $\Lambda_\infty(\omega-, f_s)$ is the asymptotic sum of the asymptotically convergent series

$$\sum_{h=0}^{\infty}{}' (-)^{h+1} f_s^{(h)}(\omega)\, \varphi_{h+1}(\omega-).$$

If for large positive x

$$f(x) \sim \sum_{h=0}^{\infty}{}' c_h (x-a)^{\sigma_h - 1} \log^{k_h}(x-a)$$

where the exponents k_h are integers $\geqq 0$, then

$$-\int_{\omega}^{H_\infty(a)} f_s(x)\,dx \sim \sum_{h=0}^{\infty}{}' c_h\, \psi(\omega - a),\ \sigma_h - s,\ k_h)$$

where ψ is defined in (3.1.5).

Proof: For large positive x the function $f_s^{(m)}(x)$ tends, according to Theorem 3.5.3, asymptotically to zero as $m \to \infty$.

If $\operatorname{Re} s$ is sufficiently large, then we have according to the primitive sum formula of Euler (1.2.1) for each positive integer m

$$
(3.6.2)\quad \left\{
\begin{aligned}
\sum_{n=1}^{\infty} f_s(n) &= \sum_{0<n<\omega} f_s(n) + \sum_{n \geq \omega} f_s(n) \\
&= \sum_{0<n<\omega} f_s(n) + \int_{\omega}^{H_\infty(a)} f_s(x)\,dx - \Lambda_m(\omega-, f_s) \\
&\qquad + (-)^{m-1} \int_{\omega}^{\infty} f_s^{(m)}(x)\,\varphi_m(x)\,dx
\end{aligned}
\right.
$$

since the first integral occurring on the right-hand side does not change its value if the upper limit $H_\infty(a)$ is replaced by ∞. If s lies in a given bounded region Δ, then the last term in (3.6.2) represents, for sufficiently large integer m, an analytic function of s in Δ. Each term occurring either in the sum $\sum_{0<n<\omega} f_s(n)$ or in the sum $\Lambda_m(\omega-, f_s)$ is an entire function of s. According to Theorem 3.4.1 the second term occurring on the right-hand side of (3.6.2) is an analytic function of s in the whole complex s-plane, the points $\sigma_0, \sigma_1, \ldots$ excepted; these points σ_h are poles and the value which the said term assumes at $s = \sigma_h$ is equal to the constant term occurring in the Laurent expansion of the function according to powers of $s - \sigma_h$. In this way we see that the right-hand side of (3.6.2) represents a function $\chi(s)$ which can be continued analytically in the whole complex s-plane, the points $\sigma_0, \sigma_1, \ldots$ excepted. Each point σ_h $(h=0, 1, \ldots)$ is a pole of $\chi(s)$.

For each s in Δ, we find, by means of (3.6.2), if m is sufficiently large,

$$
\sum_{0<n<\omega} f_s(n) = -\int_{\omega}^{H_\infty(a)} f_s(x)\,dx + \chi(s) + \Lambda_m(\omega-, f)
$$
$$
\qquad + (-)^m \int_{\omega}^{\infty} f_s^{(m)}(x)\,\varphi_m(x)\,dx
$$

which gives the required result, since the last term tends asymptotically to zero as $m \to \infty$.

And now the final remark; according to Theorem 3.4.1, $\chi(\sigma_h)$ is for $h=0, 1, \ldots$ the constant term in the Laurent expansion of $\chi(s)$ according to powers of $s-\sigma_h$.

Example 1: To determine for large positive ω the asymptotic behavior of the sum

$$\sum_{0<n<\omega} (n^2+1)^\sigma$$

where σ denotes a complex number independent of ω, we apply Theorem 3.6.1 with

$$a=0; \qquad f(x)=(x^2+1)^\sigma; \qquad f_s(x)=x^{-s}(x^2+1)^\sigma.$$

In this way we find

$$\sum_{0<n<\omega} f_s(n) \sim -\int_{\omega}^{H_\infty(0)} f_s(x)\,dx + \chi(s) + \Lambda_\infty(\omega-, x^{-s}(x^2+1)^\sigma).$$

In the whole complex s-plane, the points $2\sigma+1-2h$ ($h=0, 1, \ldots$) excepted, $\chi(s)$ is the analytic continuation of the function represented in the half-plane $\operatorname{Re} s > 1+2\operatorname{Re}\sigma$ by the series

$$\sum_{n=1}^{\infty} n^{-s}(n^2+1)^\sigma;$$

the points $2\sigma+1-2h$ ($h=0, 1, \ldots$) are poles of $\chi(s)$; $\chi(2\sigma+1-2h)$ is the constant term in the Laurent expansion of $\chi(s)$ according to powers of $s-2\sigma-1+2h$.

For $x>1$

$$f_s(x)=\sum_{h=0}^{\infty}\binom{\sigma}{h} x^{2\sigma-2h-s}$$

so that for $\omega>1$

$$-\int_{\omega}^{H_\infty(a)} f_s(x)\,dx=\sum_{h=0}^{\infty}\binom{\sigma}{h}\frac{\omega^{2\sigma-2h-s+1}}{2\sigma-2h-s+1}$$

provided that the possible term with $2\sigma-2h-s+1=0$ is replaced by $\binom{\sigma}{h}\log\omega$.

Example 2: Determine for large positive ω the asymptotic behavior of

$$\sum_{0 \leqq n < \omega} (n-a)^{\sigma-1} (1+p(n-a)^{\rho})^{\tau}$$

where $\rho > 0$, a, $p \neq 0$, σ and τ are fixed, where a is not an integer $\geqq 0$ and where

$$1 + p(n-a)^{\rho} \neq 0 \quad \text{for} \quad n = 0, 1, 2, \ldots$$

Put

$$f(x) = (1+p(x-a)^{\rho})^{\tau}.$$

According to Theorem 3.6.1 the sum is asymptotically equal to

$$(3.6.3) \qquad -\int_{\omega}^{H_{\infty}(a)} (x-a)^{\sigma-1} f(x)\, dx + \chi(1-\sigma) + \Lambda_{\infty}(\omega-, (x-a)^{\sigma-1} f(x))$$

where $\chi(s)$ is in the whole complex s-plane, the points $1+\rho\tau-\rho h$ $(h=0, 1, \ldots)$ excepted, the analytic continuation of the function represented for

$$\operatorname{Re} s > 1 + \rho \operatorname{Re} \tau$$

by

$$\sum_{n=0}^{\infty} (n-a)^{-s} (1+p(n-a)^{\rho})^{\tau}.$$

The first term occurring in (3.6.3) is for $|\omega - a| > |p|^{-1/\rho}$ equal to

$$-\sum_{h=0}^{\infty} \binom{\tau}{h} p^{\tau-h} \int_{\omega}^{H_{\infty}(a)} (x-a)^{\sigma+\rho(\tau-h)-1}\, dx = \sum_{h=0}^{\infty} \binom{\tau}{h} p^{\tau-h} \frac{(\omega-a)^{\sigma+\rho(\tau-h)}}{\sigma+\rho(\tau-h)}$$

where the possible term with

$$\sigma + \rho(\tau - h) = 0$$

must be replaced by $\binom{\tau}{h} p^{\tau-h} \log(\omega - a)$.

Theorem 3.6.2: Let a be a fixed real number. Assume that the fixed function $f(x)$ is for $x > a$ infinitely often differentiable, that, for large positive x, $f^{(m)}(x)$ tends asymptotically to zero as $m \to \infty$ and that $f(x)$ possesses at $a+$ a Hadamard expansion with exponents $\sigma_h - 1$ $(h = 0, 1, \ldots)$.

Then the function $f_s(x) = (x-a)^{-s} f(x)$ has for large positive ω the property

$$\sum_{a<n<\omega} f_s(n) \sim \int_{H_{a+}}^{\omega} f_s(x)\,dx + \chi(s) + \Lambda_\infty(\omega-, f_s)$$

where in the whole complex s-plane, the points $\sigma_0, \sigma_1, \ldots$ excepted, $\chi(s)$ is the analytic continuation of the function represented for sufficiently large Re s by

$$\sum_{n>a} f_s(n) - \int_{H_{a+}}^{\infty} f^s(x)\,dx\,. \tag{3.6.4}$$

Proof: For each $\xi > a$ in the neighborhood of a, we have according to the primitive sum formula (1.2.1) of Euler, if m is sufficiently large,

$$\sum_{a<n<\omega} f_s(n) - \int_{\xi}^{\omega} f_s(x)\,dx$$

$$= \Lambda_m(\omega-, f_s) - \Lambda_m(\xi, f_s) + (-)^{m-1} \int_{\xi}^{\infty} f_s^{(m)}(x)\,\varphi_m(x)\,dx -$$

$$-(-)^{m-1} \int_{\omega}^{\infty} f_s^{(m)}(x)\,\varphi_m(x)\,dx$$

hence

$$\sum_{a<n<\omega} f_s(n) - \int_{H_{a+}}^{\omega} f_s(x)\,dx \tag{3.6.5}$$

$$= \chi(s) + \Lambda_m(\omega-, f_s) - (-)^{m-1} \int_{\omega}^{\infty} f_s^{(m)}(x)\,\varphi_m(x)\,dx$$

where

$$\chi(s) = -\int_{H_{a+}}^{\xi} f_s(x)\,dx - \Lambda_m(\xi, f_s) + (-)^{m-1} \int_{\xi}^{\infty} f_s^{m)}(x)\,\varphi_m(x)\,dx. \tag{3.6.6}$$

Let Δ be a bounded region. Either of the last two terms represent for sufficiently large positive integer m a function of s which is analytic in Δ. The first term on the right-hand side of (3.6.6) represents according to Theorem 3.4.1 a function of s which is everywhere analytic except at the points σ_h ($h=0, 1, \ldots$). Consequently $\chi(s)$ is an analytic function of s in the whole complex s-plane, the points $\sigma_0, \sigma_1, \ldots$ excepted, whereas $\chi(\sigma_h)$ is the constant term in the Laurent expansion of $\chi(s)$ according to powers of $s-\sigma_h$.

By letting ω tend to infinity in (3.6.5) we see that $\chi(s)$ is represented for sufficiently large Re s by (3.6.4). The required result follows from (3.6.5).

Example 3: To determine for large positive ω the asymptotic behavior of the difference

$$\sum_{0<n<\omega} n^{-s}\sqrt{1+n\log n} - \int_{H_{0+}}^{\omega} x^{-s}\sqrt{1+x\log x}\,dx$$

we apply Theorem 3.6.2 with

$$a=0; \qquad f(x)=\sqrt{1+x\log x}; \qquad f_s(x)=x^{-s}\sqrt{1+x\log x}.$$

In this way we find that the said difference is asymptotically equal to

$$\chi(s)+\Lambda_\infty(\omega-, f_s)$$

where in the whole complex s-plane, the points 1, 2, ... excepted, $\chi(s)$ is the analytic continuation of the function represented in the half-plane Re $s>3/2$, the points 2, 3, ... excepted, by

$$\sum_{n=1}^{\infty} n^{-s}\sqrt{1+n\log n} - \int_{H_{0+}}^{\infty} x^{-s}\sqrt{1+x\log x}\,dx.$$

In this case we can not apply Theorem 3.6.1, since $f(x)$ does not possess a Hadamard expansion at infinity.

We have found a simple asymptotic expansion for the difference between the sum $\sum_{0<n<\omega} f_s(n)$ and the corresponding integral, but not for either of these two functions separately. To do so is not possible, since the integral does not possess a simple asymptotic expansion.

Theorem 3.6.3: *Assume that the conditions of Theorem 3.6.1 are satisfied. Assume, moreover, that for large posi-*

tive x, the fixed function $g(x) \neq 0$ is infinitely often differentiable in such a way that for large positive x and for each constant integer $h \geqq 0$

$$g^{(h)}(x) = O\, x^{-h\delta}\, g(x)$$

where δ is a positive number independent of x and h.

Assume finally that $(x-a)^{-1} g(x)$ possesses at infinity a normal Hadamard expansion with parameter a and exponents $\tau_0, \tau_1, \ldots$. Then the function

$$f_s(x) = g^{-s}(x)\, f(x)$$

has the property that for large positive ω

$$\sum_{0<n<\omega} f_s(n) \sim -\int_{\omega}^{H(\infty, a)} f_s(x)\, dx + \chi(s) + \Lambda_{\infty}(\omega-, f_s)$$

where in the whole complex s-plane, the points $\sigma_h + \tau_k$ $(h \geqq 0;\ k \geqq 0)$ excepted, $\chi(s)$ is the analytic continuation of the function represented for sufficiently large Re s by the series $\sum_{n=1}^{\infty} f_s(n)$. Each point $\sigma_h + \tau_k$ is a pole of this function.

Proof: Applying Theorem 3.5.3 we see that $f_s(x)$ is for large positive x infinitely often differentiable and that $f_s^{(m)}(x)$ tends asymptotically to zero as $m \to \infty$. According to Theorem 3.3.2 the function $(x-a)^s g^{-s}(x)$ possesses at infinity a Hadamard expansion with parameter a and exponents $\sigma_h + \tau_k$ $(h \geqq 0\ ;\ k \geqq 0)$. The proof given in Theorem 3.6.1, the final remark excepted, can here be repeated and this gives the required result.

Example 4: In order to determine again for large positive ω the asymptotic behavior of the sum

$$\sum_{0<n<\omega} (n^2+1)^{\sigma}$$

treated already in Example 1 of this section, we can apply Theorem 3.6.3 with

$$a = 0; \quad f(x) = 1; \quad g(x) = (x^2+1)^{1/2}; \quad f_s(x) = g^{-s}(x).$$

The function $x^{-1} g(x) = (1+x^{-2})^{1/2}$ possesses at infinity a normal Hadamard

expansion with parameter $a = 0$ and with exponents $0, -2, -4, \ldots$. In this way we find

$$(3.6.7) \qquad \sum_{0 \leq n < \omega} (n^2+1)^{-s/2} \sim - \int_{\omega}^{H_\infty(0)} (x^2+1)^{-s/2}\, dx + \chi(s)$$
$$+ \Lambda_\infty(\omega -, (x^2+1)^{-s/2})$$

where in the whole complex s-plane, the points $1 - 2m$ $(m = 0, 1, \ldots)$ excepted, $\chi(s)$ is the analytic continuation of the function defined in the half-plane $\operatorname{Re} s > 1$ by the series $\sum_{n=1}^{\infty} (n^2+1)^{-s/2}$.

For $x > 1$

$$f_s(x) = \sum_{h=0}^{\infty} \binom{-\tfrac{1}{2}s}{h} x^{-s-2h}$$

so that

$$(3.6.8) \qquad - \int_{\omega}^{H_\infty(0)} (x^2+1)^{-s/2}\, dx = \sum_{h=0}^{\infty}{}' \binom{-\tfrac{1}{2}s}{h} \frac{\omega^{1-s-2h}}{1-s-2h}$$

where the possible term with $1-s-2h=0$ must be replaced by $\binom{-\tfrac{1}{2}s}{h} \log \omega$.

If m is an integer ≥ 0, then we can not expect that $\chi(1-2m)$ is the constant term in the Laurent expansion of $\chi(s)$ according to powers of $s - 1 + 2m$. There is a general rule, formulated and proved in Section 7.5, which enables us to calculate the difference between $\chi(1-2m)$ and this constant term, but here we do not need this rule. The last term in (3.6.7) is the asymptotic expansion of a function which is analytic at the point $s = 1 - 2m$. Using (3.6.8) we see that the first term occurring on the right-hand side of (3.6.7) is equal to

$$(3.6.9) \qquad \binom{-\tfrac{1}{2}s}{m} \frac{\omega^{1-s-2m}}{1-s-2m}$$

plus a function which is analytic at $s = 1 - 2m$. If we put

$$\lim_{s \to 1-2m} (s - 1 + 2m)^{-1} \left\{ \binom{-\tfrac{1}{2}s}{m} - \binom{m - \tfrac{1}{2}}{m} \right\} = - v_m$$

then (3.6.9) is equal to

$$\binom{m-\frac{1}{2}}{m}\frac{1}{1-s-2m}+\binom{m-\frac{1}{2}}{m}\log\omega+v_m+o(1)$$

where $o(1)$ denotes a function of s which is analytic and equal to zero at $s=1-2m$. The term (3.6.9) augmented by $\chi(s)$ yields a function which is analytic at $1-2m$, since all the other terms are analytic at that point. Consequently

$$\chi(s)=-\binom{m-\frac{1}{2}}{m}\frac{1}{1-s-2m}+\gamma_m+o(1)$$

where γ_m is the constant term occurring in the Laurent expansion of $\chi(s)$ according to powers of $s-1+2m$. Substitution in (3.6.7) gives therefore for $s=1-2m$

$$\sum_{0<n<\omega}(n^2+1)^{-s/2}\sim\binom{m-\frac{1}{2}}{m}\log\omega+v_m+\gamma_m+\sum_{\substack{h=0\\ h\neq m}}^{\infty}\binom{-\frac{1}{2}s}{h}\frac{\omega^{1-s-2h}}{1-s-2h}$$

$$+\Lambda_\infty(\omega-,\ (x^2+1)^{-s/2}).$$

This yields the asymptotic behavior of the sum $\sum\limits_{0<n<\omega}(n^2+1)^{-s/2}$ for the exceptional values $s=1,-1,-3,\ldots$.

Theorem 3.6.4: *Assume that the conditions of Theorem 3.6.2 are satisfied. Assume, moreover, that the fixed function $g(x)$ is, for $x>a$, infinitely often differentiable, that $(x-a)^{-1}\,g(x)$ possesses at $a+$ a normal Hadamard expansion with exponents $\tau_0, \tau_1, \ldots$ and that for large positive x and for each constant integer $h\geqq 0$*

$$g^{(h)}(x)=O\,x^{-h\delta}\,g(x)$$

where δ is a positive number independent of x and h.

Then the function

$$f_s(x)=g^{-s}(x)\,f(x)$$

has the property that for large positive ω

$$\sum_{a<n<\omega}f_s(n)\sim\int_{H_{a+}}^{\omega}f_s(x)\,dx+\chi(s)+\Lambda_\infty(\omega-,\ f_s)$$

where in the whole complex s-plane, the points $\sigma_h + \tau_k$ ($h \geqq 0$; $k \geqq 0$) excepted, $\chi(s)$ is the analytic continuation of the function represented for sufficiently large Res by

$$\sum_{n>a} g^{-s}(n)\, f(n) - \int_{H_{a+}}^{\infty} g^{-s}(x)\, f(x)\, dx.$$

Each point $\sigma_h + \tau_k$ is a pole of this function.

The proof is the same as in Theorem 3.6.2.

3.7. Analytic continuation for residues.

Theorem 3.7.1: Conditions: (1) Let a be a real number. (2) $g(x) \neq 0$ and $f(x)$ are continuous for $x > a$. (3) For sufficiently large positive x the functions $f(x)$ and $g(x)$ are infinitely often differentiable, $f^{(m)}(x)$ tends asymptotically to zero for $m \to \infty$, $g(x)$ and $g^{-1}(x)$ are asymptotically finite, arg $g(x)$ is bounded; and for each fixed positive integer $h \geqq 0$ the order relation

$$g^{(h)}(x) = O\, x^{-h\delta}\, g(x) \tag{3.7.1}$$

holds, where $\delta > 0$ is independent of x and h.

Assertion: (1) If $f(x)$ and $g(x)$ are continuous at a with $g(a) \neq 0$, then $r(a, P, g^{-s}(x)\, f(x))$ is an entire function of s.

(2) If $f(x)$ possesses at $a+$ a Hadamard expansion with exponents $\sigma_h - 1$ ($h = 0, 1, \ldots$) and if $(x-a)^{-1}\, g(x)$ possesses at $a+$ a normal Hadamard expansion with exponents $\tau_0, \tau_1, \ldots$, then $r(H_{a+}, P, g^{-s}(x)\, f(x))$ is an analytic function $\chi(s)$ of s in the whole complex s-plane, the points $\sigma_h + \tau_k$ ($h \geqq 0$, $k \geqq 0$) excepted. These points $\sigma_h + \tau_k$ are poles of the function.

In the particular case that $g(x) = x - a$, we find that $r(H_{a+}, P, (x-a)^{-s}\, f(x))$ is an analytic function $\chi_1(s)$ of s, the points $\sigma_0, \sigma_1, \ldots$ excepted. These points are poles of $\chi_1(s)$. Each σ occurring in the sequence $\sigma_0, \sigma_1, \ldots$ has the property that $\chi_1(\sigma)$ is the constant term in the Laurent expansion of $\chi_1(s)$ according to powers of $s - \sigma$; the function has at

the points $s \neq \sigma$ in the neighborhood of σ the property

$$\chi_1(s) = \sum_{\substack{h \\ \sigma_h = \sigma}} k_h!\, c_h (s-\sigma)^{-k_h-1} + \chi_1(\sigma) + o(1)$$

where $o(1)$ denotes a function of s which is analytic and equal to zero at $s=\sigma$; the numbers σ_h and k_h are the numbers occurring in the Hadamard expansion (3.4.2).

Remark: In the general case we shall find in Section 7.5 a similar, but slightly more complicated result for the values of $\chi(\sigma)$ where σ occurs in the sequence $\sigma_h + \tau_k$ ($h \geqq 0$; $k \geqq 0$).

Proof of assertion (1): Assume that s lies in a bounded region Δ. According to Theorem 3.5.3 the conditions of Theorem 1.5.1 are satisfied with $f(x)$ replaced by $f_s(x) = g^{-s}(x) f(x)$, so that for sufficiently large ξ

$$(3.7.2) \qquad r(a, P, f_s) = r(a, \xi, f_s) - \Lambda_m(\xi-, f_s) + (-)^{m-1} \int_\xi^\infty f_s^{(m)}(x) \varphi_m(x)\, dx$$

if m is sufficiently large. According to the definition given in (1.5.4)

$$(3.7.3) \qquad r(a, \xi, f_s) = \sum_{a<n<\xi} f_s(n) - \int_a^\xi f_s(x)\, dx.$$

If m is sufficiently large, then each term occurring on the right-hand sides of (3.7.2) and (3.7.3) represents a function of s which is analytic in Δ. This completes the proof of assertion (1).

Proof of assertion (2): If $\beta > a$ is chosen in such a way that the interval $a < x \leqq \beta$ does not contain an integer, then $r(\beta, P, f_s)$ is according to assertion (1) an entire function of s. Furthermore it follows from the definition given in (1.5.4), that

$$r(H_{a+}, P, f_s) = r(\beta, P, f_s) - \int_{H_{a+}}^{\beta} f_s(x)\, dx$$

so that this function possesses according to Theorem 3.4.1 the required properties.

In the same way we find the following theorem.

Theorem 3.7.2: Conditions: (1) Let a be a real number. (2) $g(x) \neq 0$ and $f(x)$ are continuous for $x < a$. (3) For negative x with sufficiently large $|x|$ the functions $f(x)$ and $g(x)$ are infinitely often differentiable, $f^{(m)}(x)$ tends asymptotically to zero for $m \to \infty$, $g(x)$ and $g^{-1}(x)$ are asymptotically finite, $\arg g(x)$ is bounded and the order relation (3.7.1) holds for each fixed integer $h \geqq 0$.

Assertion: (1) If $f(x)$ and $g(x)$ are continuous at a with $g(a) \neq 0$, then $r(a, -P, g^{-s}(x) f(x))$ is an entire function of s.

(2) If $f(x)$ possesses at $a-$ a Hadamard expansion with exponents $\sigma_h - 1$ ($h=0, 1, \ldots$) and if $(a-x)^{-1} g(x)$ possesses at $a-$ a normal Hadamard expansion with exponents $\tau_0, \tau_1, \ldots$, then $r(H_{a-}, -P, g^{-s}(x) f(x))$ is an analytic function of s in the whole complex s-plane, the points $\sigma_h + \tau_k$ ($h \geqq 0$; $k \geqq 0$) excepted. These points $\sigma_h + \tau_k$ are poles of the function.

In the particular case $g(x) = a - x$ we find that the residue $r(H_{a-}, -P, (a-x)^{-s} f(x))$ is an analytic function $\chi_1(s)$ of s, the points $\sigma_0, \sigma_1, \ldots$ excepted. These points are poles of $\chi_1(s)$. Each σ occurring in the sequence $\sigma_0, \sigma_1, \ldots$ has the property that $\chi_1(\sigma)$ is the constant term in the Laurent expansion of $\chi_1(s)$ according to powers of $s-\sigma$; the function has at the points $s \neq \sigma$ in the neighborhood of σ the property

$$\chi_1(s) = \sum_h k_h!\ c_h (s-\sigma)^{-k_h-1} + \chi_1(\sigma) + o(1)$$

where $o(1)$ denotes a function of s which is analytic and equal to zero at $s = \sigma$; the numbers σ_h and k_h are the numbers occurring in the Hadamard expansion (3.4.4).

Theorem 3.7.3: Assume that the conditions of Theorem 3.7.1 are satisfied and that $f(x)$ and $g(x)$ are continuous at a with $g(a) \neq 0$. Assume that b is an arbitrary complex number such that $b - a$ is not $\geqq 0$. If $f(x)$ possesses at infinity a Hadamard expansion with parameter b and exponents $\sigma_h - 1$ ($h = 0, 1, \ldots$) and if $(x-b)^{-1} g(x)$ possesses

at infinity a normal Hadamard expansion with parameter b and exponents τ_k ($k = 0, 1, \ldots$), then the function

$$f_s(x) = g^{-s}(x)\, f(x) \qquad (x > a)$$

has the property that

$$r(a, P, f_s) = -\int_a^{H_\infty(b)} f_s(x)\, dx + \chi(s)$$

where in the whole complex s-plane, the points $\sigma_h + \tau_k$ ($h \geqq 0$, $k \geqq 0$) excepted, $\chi(s)$ is the analytic continuation of the function represented for sufficiently large Re s by

$$\chi(s) = \sum_{n>a} f_s(n).$$

The points $\sigma_h + \tau_k$ ($h \geqq 0$; $k \geqq 0$) are poles of $\chi(s)$.

In the particular case $g(x) = x - b$ the function $\chi(s)$ is analytic in the whole complex s-plane, the points $\sigma_0, \sigma_1, \ldots$ excepted. Each point σ occurring in the sequence $\sigma_0, \sigma_1, \ldots$ is a pole of $\chi(s)$; $\chi(\sigma)$ is the constant term in the Laurent expansion of $\chi(s)$ according to powers of $s - \sigma$. More precisely,

$$\chi(s) = \sum_{\substack{h \\ \sigma_h = \sigma}} k_h!\, c_h (s-\sigma)^{-k_h - 1} + \chi(\sigma) + o(1)$$

where $o(1)$ denotes a function of s which is analytic and equal to zero at $s = \sigma$; the numbers c_h and k_h are determined by the Hadamard expansion

$$f(x) \sim \sum_{h=0}^{\infty}{}' c_h (x-b)^{\sigma_h - 1} \log^{k_h}(x-b)$$

with parameter b which the function $f(s)$ possesses at infinity; the exponents k_h are integers $\geqq 0$.

Proof: Put

$$r(a, P, f_s) + \int_a^{H_\infty(b)} f_s(x)\,dx = \chi(s).$$

The first term represents according to assertion (1) of Theorem 3.7.1 an entire function of s. The function represented by the second term is according to Theorem 3.4.3 analytic in the whole complex s-plane, the points $\sigma_h + \tau_k$ ($h \geqq 0$, $k \geqq 0$) excepted. Consequently $\chi(s)$ is an analytic function of s except at the points $s = \sigma_h + \tau_k$. According to the remark added to Theorem 1.5.1 we have for sufficiently large Re s

$$\chi(s) = \sum_{n>a} f_s(n) - \int_a^{H_\infty(b)} f_s(x)\,dx + \int_a^{H_\infty(b)} f_s(x)\,dx = \sum_{n>a} f_s(n).$$

The remark concerning the special case $g(x) = x - b$ follows from Theorem 3.4.3.

Remark: If the condition that $b - a$ is not $\geqq 0$ and that $f(x)$ and $g(x)$ are continuous at a is replaced by the weaker condition that $b - a$ is not positive and that $f_s(x)$ is integrable from an integrating neutrix I_{a+} to a number $a' > a$, then

$$r(I_{a+}, P, f_s) = -\int_{I_{a+}}^{H_\infty(b)} f_s(x)\,dx + \chi(s)$$

where $\chi(s)$ is the same function as above. Indeed we can choose $a' > a$ in such a way that the interval $a < x \leqq a'$ does not contain an integer; according to Theorem 3.7.3 applied with a replaced by a',

$$\begin{aligned} r(I_{a+}, P, f_s) &= -\int_{I_{a+}}^{a'} f_s(x)\,dx + r(a', P, f_s) \\ &= -\int_{I_{a+}}^{a'} f_s(x)\,dx - \int_{a'}^{H_\infty(b)} f_s(x)\,dx + \chi(s) \\ &= -\int_{I_{a+}}^{H_\infty(b)} f_s(x)\,dx + \chi(s) \end{aligned}$$

where $\chi(s)$ is the analytic continuation of the function represented for sufficiently large Re s by the series

$$\sum_{n>a'} f_s(n) - \sum_{n>a} f_s(n).$$

CHAPTER 4. APPLICATIONS

4.1. On some residues involving the zeta function.

Theorem 4.1.1: *For each real a and for each $s \neq 1$*

$$r(H_{a+}, P, (x-a)^{-s}) = \zeta(s, \Theta) \tag{4.1.1}$$

where $a+\Theta$ is the smallest integer $> a$ (hence $0 < \Theta \leq 1$) and where $\zeta(s, \Theta)$ is the analytic continuation of the function represented in the half-plane $\mathrm{Re}\, s > 1$ by the series

$$\zeta(s, \Theta) = \sum_{n=0}^{\infty} (n+\Theta)^{-s}. \tag{4.1.2}$$

Moreover

$$r(H_{a+}, P, (x-a)^{-1}) = \Gamma_\Theta \tag{4.1.3}$$

where Γ_Θ is the constant term in the Laurent expansion

$$\zeta(s, \Theta) = \frac{1}{s-1} + \Gamma_\Theta + \ldots \tag{4.1.4}$$

of $\zeta(s, \Theta)$ according to powers of $s-1$. In particular, if a is an integer, then $\Theta = 1$ and Γ_1 is the constant of Euler.

First proof of (4.1.1). The conditions of Theorem 3.7.1 are satisfied with $f(x) = 1$; $g(x) = x-a$; there is only one exponent σ_h, namely $\sigma_0 = 1$. Consequently $r(H_{a+}, P, (x-a)^{-s})$ is, in the whole complex s-plane, the point $s = 1$ excepted, the analytic continuation of the function represented in the half-plane Re $s > 1$ by

$$\sum_{n>a} (n-a)^{-s} - \int_{H_{a+}}^{\infty} (x-a)^{-s}\, dx = \zeta(s, \Theta).$$

Second proof of (4.1.1). The conditions of the remark added to Theorem 3.7.3 are satisfied with $f(x) = 1$; $g(x) = x - a$; $b = a$; $I_{a+} = H_{a+}$, so that

$$r(H_{a+}, P, (x-a)^{-s}) = -\int_{H_{a+}}^{H\omega(a)} (x-a)^{-s}\, dx + \chi_1(s) = \chi_1(s)$$

where $\chi_1(s)$ is, in the whole complex s-plane, the point $s = 1$ excepted, the analytic continuation of the function represented in the half-plane $\operatorname{Re} s > 1$ by

$$\sum_{n>a} (n-a)^{-s} = \zeta(s, \Theta).$$

Formula (4.1.3) follows immediately from Theorem 3.7.1, and also from Theorem 3.7.3.

Remark: According to the definition given in (2.2.3), $r(a, \xi, (x-a)^h)$ is for each $\xi > a$ and for each integer $h \geqq 0$ equal to

$$\sum_{a<n<\xi} (n-a)^h - \int_a^{\xi} (x-a)^h\, dx$$

and this is, according to the primitive sum formula (1.2.1) of Euler applied with $m = h + 1$, equal to

$$\Lambda_{h+1}(\xi-, (x-a)^h) - \Lambda_{h+1}(a+, (x-a)^h).$$

The first term is negligible in P and the second term is according to the definition given in (1.2.2) equal to $(-)^h h!\, \varphi_{h+1}(a+)$. Comparison with (4.1.1) gives therefore for $0 < \Theta \leqq 1$ and for each integer $h \geqq 0$

$$(4.1.5) \qquad (-)^h h!\, \varphi_{h+1}(a+) = \zeta(-h, \Theta).$$

This result indicates one of the advantages which the neutralized sum formula of Euler (Theorem 2.5.6) has over the primitive formula. The latter involves the coefficients $\varphi_{h+1}(a+)$, therefore the coefficients $\zeta(-h, \Theta)$ and may therefore be appropriate in the investigation of a function which possesses an asymptotic expansion in which each coefficient is, apart from an elementary factor, equal to $\zeta(-h, \Theta)$, where h is an integer $\geqq 0$. For instance the function $\sum_{0 \leqq n < \omega} (\omega+n)^{-s}$ treated in Section 1.2 is such a function, but the

class formed by the functions with this property is very restricted. The primitive formula of Euler is certainly not convenient if the coefficients occurring in the asymptotic expansion are not so simple as that, for instance if they involve $\zeta(s)$, where s is not an integer $\leqq 0$ and also if in their definition we need the function represented in the half-plane $\operatorname{Re} s > 1$ by $\sum_{n=1}^{\infty} (n^2+1)^{-s/2}$.

The next theorem gives a more general result.

Theorem 4.1.2. *For each real a, for each integer $k \geqq 0$ and for each $s \neq 1$*

$$r(H_{a+}, P, (x-a)^{-s} \log^k (x-a)) = (-)^k \zeta^{(k)}(s, \Theta)$$

where $a+\Theta$ is the smallest integer $> a$ and where $\zeta^{(k)}(s, \Theta)$ is the k^{th} derivative with respect to s of the analytic continuation $\zeta(s, \Theta)$ of the function represented in the half-plane $\operatorname{Re} s > 1$ by (4.1.2).

Proof: Applying Theorem 3.7.1 with

$$f(x) = \log^k (x-a); \qquad g(x) = 1; \qquad I_{a+} = H_{a+}; \qquad \sigma_0 = 1$$

and using the fact that

$$\int_{H_{a+}}^{H_{+}(a)} (x-a)^{-s} \log^k (x-a)\, dx = 0$$

we see that $r(H_{a+}, P, (x-a)^{-s} \log^k (x-a))$ is the analytic continuation of the function which is represented in the half-plane $\operatorname{Re} s > 1$ by

$$\sum_{n>a} (n-a)^{-s} \log^k (n-a) = (-)^k \zeta^{(k)}(s, \Theta).$$

4.2. On $\sum_{n>0} n^{\sigma-1} \left(\omega + n \log \dfrac{n}{\omega}\right)^{\tau}$.

Determine for large positive ω the asymptotic behavior of the sum

$$\sum_{n=1}^{\infty} j(n), \qquad \text{where} \quad j(x) = x^{\sigma-1} \left(\omega + x \log \frac{x}{\omega}\right)^{\tau};$$

σ and τ denote fixed complex numbers with $\operatorname{Re}(\sigma+\tau) < 0$; this inequality insures the convergence.

The function $u(x)=\omega+x\log\frac{x}{\omega}$ has for $\omega=px$ the property

$$u(x)=(p-\log p)\,x\geqq x,$$

$$|u'(x)|=|1+\log p|=\frac{|1+\log p|}{p-\log p}\,u(x)\,x^{-1}=O\,u(x)\,x^{-1}$$

and for each fixed integer $h\geqq 2$

$$u^{(h)}(x)=(-)^h\,(h-2)!\;x^{1-h}=O\,u(x)\,x^{-h}.$$

According to Theorem 3.5.2, applied with $\delta=1$, the function

$$g(x)=\left(\omega+x\log\frac{x}{\omega}\right)^{\tau}$$

satisfies therefore, for large positive x and for each fixed integer $h\geqq 0$, the order relation

$$g^{(h)}(x)=O\,x^{-h}\,g(x)$$

uniformly in ω. Consequently the function $j(x)=x^{\sigma-1}\,g(x)$ satisfies the order relation

$$j^{(h)}(x)=O\,x^{-h}\,j(x)\quad(h\quad 0,\,1,\,\ldots).$$

The conditions of Theorem 2.2.1 are therefore satisfied with f replaced by j and with $\alpha=\omega^{\delta}$; $\beta=\infty$, where δ denotes a fixed positive number <1. We may therefore apply Theorem 2.2.2, and this yields

$$\text{(4.2.1)}\qquad \sum_{n=1}^{\infty} j(n)-\int_{H_{0+}}^{\infty} j(x)\,dx=\lim_{\xi\to\infty} r(H_{0+},\,\xi,\,j)\sim r(H_{0+},\,Q,\,j).$$

According to Theorem 2.1.1 the residue $r(H_{0+},\,Q,\,j)$ does not change its asymptotic behavior if Q is replaced by the periodic asymptotic neutrix Q^* with domain $\alpha<\xi<2\alpha$ and in this interval we have

$$\text{(4.2.2)}\qquad j(x)\sim\sum_{h=0}^{\infty}{}'\binom{\tau}{h}\,\omega^{\tau-h}\,x^{\sigma+h-1}\left(\log\frac{x}{\omega}\right)^h$$

$$\sim\sum_{h\geqq k\geqq 0}{}'\,(-)^k\binom{\tau}{h}\binom{h}{k}\,\omega^{\tau-h}(\log\omega)^k\,x^{\sigma+h-1}\,\log^{h-k}x$$

so that according to Theorem 2.3.3

$$r(H_{0+}, Q^*, f) \sim \sum_{h \geqq k \geqq 0}{}' (-)^h \binom{\tau}{h} \binom{h}{k} \omega^{\tau-h} (\log \omega)^k r(H_{0+}, Q^*, x^{\sigma+h-1} \log^{h-k} x).$$

According to Theorem 2.2.3 the residue $r(H_{0+}, Q^*, x^{\sigma+h-1} \log^{h-k} x)$ does not change its asymptotic behavior if Q^* is replaced by the periodic neutrix P, so that this residue is, according to Theorem 4.1.2, applied with $a=0$ (hence $\Theta=1$) asymptotically equal to

$$(-)^{h-k} \zeta^{(h-k)}(1-\sigma-h)$$

if σ is not an integer $\leqq 0$. In this case we find therefore

$$(4.2.3) \quad r(H_{0+}, Q^*, f) \sim \sum_{h \geqq k \geqq 0}{}' (-)^h \binom{\tau}{h} \binom{h}{k} \zeta^{(k-h)}(1-\sigma-h) \omega^{\tau-h} \log^k \omega.$$

If σ is an integer $\leqq 0$, then according to Theorem 4.1.2 we must replace in each term with $h=-\sigma$ the factor $\zeta^{(h-k)}(1-\sigma-h)$ by $\Gamma_{1,k-h}$, where $\Gamma_{10}, \Gamma_{11}, \ldots$ are the coefficients occurring in the Laurent expansion of

$$\zeta(s) = \frac{1}{s-1} + \Gamma_{10} + \Gamma_{11} \frac{(s-1)}{1!} + \Gamma_{12} \frac{(s-1)^2}{2!} + \ldots$$

according to powers of $s-1$.

Now we must determine the asymptotic behavior of the integral occurring in (4.2.1).

The substitution $x = y\omega$ gives for each positive ε

$$(4.2.4) \qquad \int_{\varepsilon\omega}^{\infty} j(x)\, dx = \omega \int_{\varepsilon}^{\infty} j(\omega y)\, dy.$$

If ε is sufficiently small, then we have in the interval $0 \leqq x \leqq \varepsilon\omega$

$$(4.2.5) \qquad j(x) = \sum_{0 \leq h \leq -Re\,\sigma} \binom{\tau}{h} \omega^{\tau-h} x^{\sigma+h-1} \left(\log \frac{x}{\omega}\right)^h + \rho(x)$$

where $\rho(x)$ is integrable from 0 to $\varepsilon\omega$. If σ is not an integer $\leqq 0$, then $\sigma+h \neq 0$ for each integer $\geqq 0$, so that according to (3.1.7) applied with $a=0$ and $p=\omega^{-1}$

$$\int_{H_{0+}}^{\varepsilon\omega} x^{\sigma+h-1} \left(\log \frac{x}{\omega}\right)^h dx = \omega^{\sigma+h} \int_{H_{0+}}^{\varepsilon} y^{\sigma+h-1} \log^h y\, dy$$

hence

$$\int_{H_{0+}}^{i\omega} j(x)\,dx = \omega \int_{H_{0+}}^{i} j(\omega y)\,dy$$

so that by (4.2.4)

$$\int_{H_{0+}}^{\infty} j(x)\,dx = \omega \int_{H_{0+}}^{\infty} j(\omega y)\,dy = \omega^{\sigma+\tau} \int_{H_{0+}}^{\infty} y^{\sigma-1}(1+y\log y)^{\tau}\,dy.$$

However, if σ is an integer $\leqq 0$, then the sum occurring in (4.2.5) contains a term with $h = -\sigma$ and this h yields according to (3.1.8), applied with $a = 0$, $p = \omega^{-1}$ and $k = -\sigma$ the supplementary term

$$(-)^{\sigma}(1-\sigma)^{-1}(\log\omega)^{1-\sigma}\binom{\tau}{-\sigma}\omega^{\sigma+\tau}$$

so that then

$$\int_{H_{0+}}^{\infty} j(x)\,dx = \omega^{\sigma+\tau}\int_{H}^{\infty} y^{\sigma-1}(1+y\log y)^{\tau}\,dy \tag{4.2.6}$$

$$+(-)^{\sigma}(1-\sigma)^{-1}\binom{\tau}{-\sigma}\omega^{\sigma+\tau}(\log\omega)^{1-\sigma}.$$

Consequently the integral under consideration is in the case that σ is not an integer $\leqq 0$ equal to $\omega^{\sigma+\tau}$ times a fixed number, but if σ is an integer $\leqq 0$, then we obtain a supplementary term involving logarithmic factors; this supplementary term has even for large positive ω a higher order of magnitude than the original term.

Example 2. Determine for large positive ω the asymptotic behavior of

$$\sum_{0<n<b} n^{\sigma-1}\left(\omega + n\log\frac{n}{\omega}\right)^{\tau}$$

where b denotes a number $\geqq \omega^{\varepsilon}$ depending on ω and where σ, τ and ε denote fixed numbers with $0<\varepsilon<\frac{1}{2}$. We shall restrict ourselves to the case that either $b \leqq \omega^{1-\varepsilon}$ or $b = c\omega$, where c is a fixed positive number.

The function

$$f(x) = x^{\sigma-1}\left(\omega + x \log \frac{x}{\omega}\right)^{\tau}$$

has according to the neutralized sum formula (Theorem 2.5.6) the property

$$\sum_{0<n<b} f(n) \sim \int_{H_{0+}}^{b} f(x)\,dx + r(H_{0+}, Q, f) + r(b, Q, f) \tag{4.2.7}$$

where Q is the periodic neutrix with domain $\alpha < \xi < 2\alpha$; here $\alpha = \omega^{\delta}$, where δ is a fixed positive number $< \varepsilon$. According to Theorem 2.5.4

$$r(b, Q, f) \sim \Lambda_{\infty}(b-, f)$$

which yields the asymptotic behavior of $r(b, Q, f)$. Furthermore, (4.2.3) gives the asymptotic behavior of $r(H_{0+}, Q, f)$, so that it is sufficient to determine the asymptotic behavior of the integral occurring in (4.2.7). In the same way as in the preceding example we see that this integral can be written in the form

$$\omega^{\sigma+\tau} \int_{H_{0+}}^{\omega^{-1}b} y^{\sigma-1}(1 + y \log y)^{\tau}\,dy \tag{4.2.8}$$

if σ is not an integer $\leqq 0$. If σ is an integer $\leqq 0$, then we obtain the same additional term as in (4.2.6).

If $b = c\omega$, then (4.2.8) is equal to

$$\omega^{\sigma+\tau} \int_{H_{0+}}^{c} y^{\sigma-1}(1 + y \log y)^{\tau}\,dy \tag{4.2.9}$$

and therefore equal to $\omega^{\sigma+\tau}$ times a fixed number. If $b = O\,\omega^{1-\varepsilon}$, then (4.2.8) is equal to

$$\omega^{\sigma+\tau} \sum_{h=0}^{\infty} \binom{\tau}{h} \int_{H_{0+}}^{\omega^{-1}b} y^{\sigma+h-1} \log^{h} y\,dy$$

$$= \omega^{\sigma+\tau} \sum_{h=0}^{\infty} \binom{\tau}{h} \psi(\omega^{-1}b, \sigma + h, h)$$

where ψ is defined by (3.1.5); the last series is not only convergent, but also asymptotically convergent.

Remark: In the case $b = c\omega$ the result obtained above yields the asymptotic behavior of the sum $\sum_{0<n<b} f(n)$, provided that the integral occurring in (4.2.9) is known. Conversely our result enables us to evaluate this integral with any desired degree of accuracy by applying the formula with a sufficiently large positive ω.

4.3. On $\sum_{0<n<2\omega} (n(\omega-n)(2\omega-n))^{1/3}$.

Theorem 4.3.1: For large positive ω

$$\sum_{0<n<2\omega} (n(\omega-n)(2\omega-n))^{1/3}$$

$$\sim 2^{1/3} \sum_{h=0}^{\infty}{}' c_h\, \omega^{2/3-h} (\zeta(-h-\tfrac{1}{3}) - \zeta(-h-\tfrac{1}{3}, \vartheta))$$

$$+ \sum_{h=0}^{\infty}{}' (-)^h \binom{1/3}{h} \omega^{1/3-2h} (\zeta(-2h-\tfrac{1}{3}, \Theta) - \zeta(-2h-\tfrac{1}{3}, 1-\Theta))$$

where $\omega - \Theta$ is the largest integer $< \omega$, where $2\omega - \vartheta$ is the largest integer $< 2\omega$ and where

$$c_h = (-)^h \sum_{k=0}^{h} 2^{-k} \binom{1/3}{k} \binom{1/3}{h-k};$$

if ω is an integer, then we must replace $\zeta(-2h-\frac{1}{3}, 1-\Theta)$ by $\zeta(-2h-\frac{1}{3})$, so that the last series drops out.

Proof: Repeated application of the neutralized sum formula of Euler (Theorem 2.5.6) gives for $f(x) = (x(\omega-x)(2\omega-x))^{1/3}$

$$\sum_{0<n<2\omega} f(n) \tag{4.3.1}$$

$$\sim \int_0^{2\omega} f(x)\,dx + r(H_{0+}, Q_1, f) + r(H_{\omega-}, Q_2, f)$$

$$+ r(H_{\omega+}, Q_3, f) + r(H_{2\omega-}, Q_4, f);$$

Q_1, Q_2, Q_3, Q_4 are the periodic asymptotic neutrices respectively with domains

$\alpha < x < 2\alpha$; $\omega - 2\alpha < x < \omega - \alpha$; $\omega + \alpha < x < \omega + 2\alpha$; $2\omega - 2\alpha < x < 2\omega - \alpha$,

where $\alpha = \omega^{\delta}$ and δ denotes a fixed positive number < 1.

The integral occurring in (4.3.1) vanishes since $f(2\omega-x) = -f(x)$. Putting $x = \omega y$ we obtain in the interval $0 \leqq x \leqq 2\alpha$

$$f(x) = 2^{1/3}\, \omega y^{1/3}\, (1-y)^{1/3} \left(1-\frac{y}{2}\right)^{1/3} \sim 2^{1/3}\, \omega \sum_{h=0}^{\infty}{}' c_h\, y^{h+1/3}$$

so that according to Theorem 2.3.3

$$r(H_{0+}, Q_1, f) \sim 2^{1/3} \sum_{h=0}^{\infty}{}' c_h\, \omega^{?/3-h}\, r(H_{0+}, Q_1, x^{h+1/3}).$$

According to the Theorems 2.1.1 and 2.2.3 we may replace on the right-hand side Q_1 by P and therefore (according to Theorem 4.1.1) $r(H_{0+}, Q_1, x^{h+1/3})$ by $\zeta(-h-\frac{1}{3})$. Treating the three other residues occurring in (4.3.1) in a similar way we obtain the required result.

4.4. Beta integrals.

Theorem 4.4.1: If ω is a large positive integer and if Θ, ϑ, σ and τ are fixed, where $0 < \Theta \leqq 1$ and $0 < \vartheta \leqq 1$ and where neither σ nor τ is an integer < 0, then

$$\sum_{n=0}^{\omega} (n+\Theta)^{\sigma}\, (\omega-n+\vartheta)^{\tau} \tag{4.4.1}$$

$$\sim \frac{\sigma!\ \tau!}{(\sigma+\tau+1)!} (\omega+\Theta+\vartheta)^{\sigma+\tau+1} + \sum_{h=0}^{\infty}{}' (-)^h \binom{\tau}{h} (\omega+\Theta+\vartheta)^{\tau-h}\, \zeta(-\sigma-h, \Theta)$$

$$+ \sum_{h=0}^{\infty}{}' (-)^h \binom{\sigma}{h} (\omega+\Theta+\vartheta)^{\sigma-h}\, \zeta(-\tau-h, \vartheta).$$

Proof: According to the neutralized sum formula of Euler (Theorem 2.5.6) applied with

$$a = -\Theta; \quad b = \omega + \vartheta; \quad f(x) = (x+\Theta)^{\sigma}\, (\omega-x+\vartheta)^{\tau}$$

the sum $\sum_{0 \leqq n \leqq \omega} f(n)$ is asymptotically equal to

$$\int_{H_{-\Theta+}}^{H_{\omega+\vartheta-}} f(x)\, dx + r(H_{-\Theta+}, Q, f) + r(H_{\omega+\vartheta-}, Q^*, f).$$

Here Q and Q^* are the periodic asymptotic neutrices respectively with domains $\alpha < x < \beta$ and $\alpha^* < \xi^* < \beta^*$, where

$$\alpha = \omega^\delta; \quad \beta = 2\omega^\delta; \quad \alpha^* = \omega - 2\omega^\delta; \quad \beta^* = \omega - \omega^\delta;$$

here δ denotes a fixed positive number < 1. In the interval $-\Theta < x < \beta$ we have

$$f(x) \sim \sum_{h=0}^{\infty}{}' (-)^h \binom{\tau}{h} (\omega + \Theta + \vartheta)^{\tau-h} (x + \Theta)^{\sigma+h}.$$

This gives, as we have seen already several times, for instance in Section 4.3

$$(4.4.2) \qquad r(H_{-\Theta+}, Q, f) \sim \sum_{h=0}^{\infty}{}' (-)^h \binom{\tau}{h} (\omega + \Theta + \vartheta)^{\tau-h} \zeta(-\sigma-h, \Theta).$$

Interchanging σ and τ, Θ and ϑ we obtain

$$r(H_{\omega+\vartheta-}, Q^*, f) \sim \sum_{h=0}^{\infty}{}' (-)^h \binom{\sigma}{h} (\omega+\Theta+\vartheta)^{\sigma-h} \zeta(-\tau-h, \vartheta).$$

The formula

$$(4.4.3) \qquad \int_{H_{-\Theta+}}^{H_{\omega+\vartheta-}} f(x)\, dx = \frac{\sigma!\ \tau!}{(\sigma+\tau+1)!} (\omega + \Theta + \vartheta)^{\sigma+\tau+1}$$

is obvious if both Re s and Re τ are > -1, since in this case the integral is the convergent beta-integral from $-\Theta$ to $\omega+\vartheta$. According to the Theorems 3.4.1 and 3.4.2, the integral occurring in (4.4.3) represents in the whole complex σ-plane, the points $-1, -2, \ldots$ excepted, an analytic function of σ and also in the whole complex τ-plane, the points $-1, -2, \ldots$ excepted, an analytic function of τ. Consequently (4.4.3) holds for each σ which is not a negative integer and for each τ which is not a negative integer. This completes the proof.

Theorem 4.4.2: *If ω is a large positive integer, if $0 < \Theta \leq 1$, $0 < \vartheta \leq 1$, l is a positive integer and τ is a fixed number which is not a negative integer, then*

$$\sum_{n=0}^{\omega} (n+\Theta)^{-l} (\omega - n + \vartheta)^{\tau}$$

$$\sim (-)^{l-1} (\omega+\Theta+\vartheta)^{\tau+1-l} (n\Gamma_\Theta + u \log(\omega+\Theta+\vartheta) - v)$$

$$+ \sum_{\substack{h=0 \\ h \neq l-1}}^{\infty}{}' (-)^h \binom{\tau}{h} \zeta(l-h, \Theta) (\omega+\Theta+\vartheta)^{\tau-h}$$

$$+ \sum_{h=0}^{\infty}{}' (-)^h \binom{-l}{h} \zeta(-\tau-h, \vartheta) (\omega+\Theta+\vartheta)^{-l-h}$$

where Γ_Θ is the constant defined in (4.1.4),

$$u = \binom{\tau}{l-1} \text{ and } v = \tau!\left(\frac{\partial}{\partial s}\frac{(-s)!}{(\tau+1-l-s)!\,(1+s)(2+s)\ldots(l-1+s)}\right)_{s=0}.$$

Proof: If $\sigma = -l$, then the integral (4.4.3) is equal to the constant term in the Laurent expansion according to powers of s of the function

$$\frac{(-l-s)!\;\tau!}{(-l-s+\tau+1)!}(\omega+\Theta+\vartheta)^{-l+\tau+1-s}.$$

This function is equal to

$$\frac{(-)^l}{s}\frac{(-s)!\quad\tau!}{(-l-s+\tau+1)!\,(s+1)\,(s+2)\ldots(s+l-1)}(\omega+\Theta+\vartheta)^{-l+\tau+1-s}$$

$$=\frac{(-)^l}{s}(u+vs-us\log(\omega+\Theta+\vartheta))\,(\omega+\Theta+\vartheta)^{-l+\tau+1}+o(1)$$

where $o(1)$ denotes a function of s which is analytic and equal to zero at $s=0$. In this way we obtain

$$\int_{H_{-\Theta+}}^{H_{\omega+\vartheta-}} f(x)\,dx = (-)^{l-1}(u\log(\omega+\Theta+\vartheta)-v)\,(\omega+\Theta+\vartheta)^{\tau+1-l}. \tag{4.4.4}$$

Consequently (4.4.1) remains true if we replace the first term on the right-hand side by (4.4.4) and if we replace the factor $\zeta(-\sigma-h,\Theta)$ with $h=l-1$ occurring in the first sum on the right-hand side of (4.4.1) by Γ_Θ. This gives the required result.

Remark: For the treatment of the case that τ is a fixed negative integer and σ is a fixed number which is not a negative integer, we have only to interchange σ and τ, Θ and ϑ. The case that both σ and τ are fixed negative integers can be treated in the same way.

4.5. Bessel functions.

Theorem 4.5.1: For large positive ω the sum

$$\sum_{-\omega<n<\omega}(\omega^2-n^2)^{\sigma-1/2}\cos\frac{un}{\omega},$$

where the real number u and the complex number σ are

fixed and where $\sigma-\frac{1}{2}$ is not a negative integer, is asymptotically equal to

$$(4.5.1)\quad \sqrt{\pi}\, 2^{\sigma} (\sigma - \tfrac{1}{2})!\; u^{-\sigma} J_{\sigma}(u)\, \omega^{2\sigma} + 2 \sum_{h=0}^{\infty}{}' \zeta(\tfrac{1}{2} - \sigma - h, \Theta)\, v_h(u)\, \omega^{\sigma-\frac{1}{2}-h};$$

here $J_{\sigma}(u)$ is the Bessel function with argument u and order σ; $\omega-\Theta$ is the largest integer $<\omega$ and the functions $v_h(u)$ are defined by the expansion

$$(4.5.2)\qquad (2-t)^{\sigma-\frac{1}{2}} \cos u(1-t) = \sum_{h=0}^{\infty} v_h(u)\, t^h$$

valid at the points t in the neighborhood of 0.

If $u=0$, then $u^{-\sigma} J_{\sigma}(u)$ means the limit

$$\lim_{t\to 0} t^{-\sigma} J_{\sigma}(t) = \frac{1}{2^{\sigma}\sigma!}.$$

Proof: Putting

$$f(x) = (\omega^2 - x^2)^{\sigma-\frac{1}{2}} \cos\frac{ux}{\omega}$$

we have according to the neutralized sum formula of Euler (Theorem 2.5.6)

$$(4.5.3)\qquad \sum_{-\omega<n<\omega} f(n) \sim \int_{H_{-\omega+}}^{H_{\omega-}} f(x)\,dx + r(H_{-\omega+}, Q, f) + r(H_{\omega-}, Q^*, f).$$

Here Q and Q^* are the periodic neutrices respectively with the domains $\alpha<\xi<\beta$ and $\alpha^*<\xi^*<\beta^*$, where

$$\alpha = -\omega + \omega^{\delta};\quad \beta = -\omega + 2\omega^{\delta};\quad \alpha^* = \omega - 2\omega^{\delta};\quad \beta^* = \omega - \omega^{\delta};$$

δ denotes a fixed positive number <1. The formula

$$(4.5.4)\qquad \int_{H_{-\omega+}}^{H_{\omega-}} f(x)\,dx = \sqrt{\pi}\, 2^{\sigma}(\sigma-\tfrac{1}{2})!\; u^{-\sigma} J_{\sigma}(u)\, \omega^{2\sigma}$$

is obvious for $\operatorname{Re}\sigma > \frac{1}{2}$, since then the integral is the convergent integral of $f(x)$ from $-\omega$ to ω and this integral has the value represented by the right-hand side of (4.5.4). Consequently (4.5.4) holds for each σ such that

$\sigma - \frac{1}{2}$ is not a negative integer, since the left-hand side of (4.5.4) represents according to the Theorems 3.4.1 and 3.4.2 an analytic function of σ, except at the points $-\frac{1}{2}, -\frac{3}{2}, \ldots$.

Because $f(x)$ is an even function of x, the two residues occurring in (4.5.3) are asymptotically equal. Using (4.5.2) with $t = \frac{x}{\omega} + 1$ we find by means of Theorem 2.3.3

$$r(H_{-\omega+}, Q, f) \sim \sum_{h=0}^{\infty}{}' v_h(u)\, \omega^{\sigma-\frac{1}{2}-h}\, r(H_{-\omega+}, Q, (\omega+x)^{\sigma-\frac{1}{2}+h}),$$

where the last residue is asymptotically equal to $\zeta(\frac{1}{2}-\sigma-h, \Theta)$. This completes the proof.

4.6. On $\sum\limits_{0<n<\omega} n^{\sigma-1}(\omega^{\kappa}-n^{\kappa})^{\tau-1}(\sin \pi n^{\lambda}\omega^{-\lambda})^{\mu}$. Determine for large positive ω the asymptotic behavior of the sum $\sum\limits_{0<n<\omega} f(n)$, where

$$f(x) = x^{\sigma-1}(\omega^{\kappa}-x^{\kappa})^{\tau-1}(\sin \pi x^{\lambda}\omega^{-\lambda})^{\mu} \quad (x>0) \tag{4.6.1}$$

and where the numbers $\varkappa$ and λ are positive and the numbers μ, σ and τ are complex.

According to the neutralized sum formula (Theorem 2.5.6) the sum $\sum\limits_{0<n<\omega} f(n)$ is asymptotically equal to

$$\int_{H_{0+}}^{H_{\omega-}} f(x)\,dx + r(H_{0+}, Q, f) + r(H_{\omega-}, Q^*, f); \tag{4.6.2}$$

here Q and Q^* are the periodic asymptotic neutrices respectively with domain $\alpha<\xi<\beta$ and $\alpha^*<\xi^*<\beta^*$, where

$$\alpha = \omega^{\delta};\quad \beta = 2\omega^{\delta};\quad \alpha^* = \omega - 2\omega^{\delta};\quad \beta^* = \omega - \omega^{\delta};$$

δ is a fixed positive number <1.

We use the fact that $f(x)$ possesses at the points $x<\omega$ in the neighborhood of ω an expansion according to powers of $\omega - x$ with exponents $\mu+\tau-1+h$ $(h = 0, 1, \ldots)$ and at the points $x>0$ in the neighborhood of the origin an expansion according to powers of x with exponents $\sigma+\lambda\mu-1+\varkappa h+2\lambda g$, where h and g are integers $\geqq 0$. If b denotes an arbitrary number between 0 and ω, then the integral

$$\int_{H_{0+}}^{b} f(x)\,dx$$

represents according to Theorem 3.4.1 an entire function of τ and moreover an analytic function of σ in the whole complex σ-plane, the points

$$\sigma = -\lambda\mu - \varkappa h - 2\lambda g \qquad (h \text{ and } g = 0, 1, 2, \ldots) \tag{4.6.3}$$

excepted. The integral

$$\int_{b}^{H_{\omega-}} f(x)\,dx$$

is according to Theorem 3.4.2 an entire function of σ and moreover an analytic function of τ in the whole complex τ-plane, the points

$$\tau = -\mu - h \quad (h = 0, 1, \ldots) \tag{4.6.4}$$

excepted. Consequently the integral occurring in (4.6.2) is an analytic function of σ and τ if the points (4.6.3) and (4.6.4) are left out of consideration. These exceptional values of σ and τ can be treated by means of a passage to the limit.

The formula

$$\int_{H_{0+}}^{H_{\omega-}} f(x)\,dx = \omega^{\sigma+(\tau-1)\kappa} \int_{H_{0+}}^{H_{1-}} y^{\sigma-1}(1-y^{\kappa})^{\tau-1}(\sin \pi y^{\lambda})^{\mu}\,dy \tag{4.6.5}$$

is obvious in the case $\operatorname{Re}\sigma > -\lambda \operatorname{Re}\mu$ and $\operatorname{Re}\tau > -\operatorname{Re}\mu$, since then the two integrals are convergent integrals with the required property. Considerations of analyticity show that (4.6.5) holds for each σ and each τ, the points (4.6.3) and (4.6.4) excepted, so that the integral occurring in (4.6.2) is equal to $\omega^{\sigma+(\tau-1)\kappa}$ times a fixed number which can be evaluated with any desired degree of accuracy.

To determine the residue $r(H_{0+}, Q, f)$ we write for small positive x

$$(1-x^{\kappa})^{\tau-1}\left(\frac{\sin \pi x^{\lambda}}{x^{\lambda}}\right)^{\mu} = \sum_{m} c_m x^m$$

the sum is extended over the numbers m of the form $m = h\varkappa + 2g\lambda$, where

h and g are integers $\geqq 0$. Then

$$f(x) = \sum_m c_m\, \omega^{(\tau-1)\kappa-\lambda\mu-m}\, x^{\sigma-1+\lambda\mu+m}$$

hence

$$r(H_{0+}, Q, f) \sim \sum_m' c_m\, \zeta(1-\sigma-\lambda\mu-m)\, \omega^{(\tau-1)\kappa-\lambda\mu-m}.$$

To determine the residue $r(H_{\omega-}, Q^*, f)$ we write for small positive y

$$(1-y)^{\sigma-1}\left(\frac{1-(1-y)^\kappa}{y}\right)^{\tau-1}\left(\frac{\sin\pi(1-y)^\lambda}{y}\right)^\mu = \sum_{h=0}^{\infty}\gamma_h y^h$$

which gives for the points $x<\omega$ in the neighborhood of ω the expansion

$$f(x) = \sum_{h=0}^{\infty}\gamma_h\, \omega^{\sigma-\mu+(\tau-1)\kappa-\tau-h}\,(\omega-x)^{\tau+\mu+h-1}$$

so that

$$r(H_{\omega-}, Q^*, f) \sim \sum_{h=0}^{\infty}{}'\, \gamma_h\, \omega^{\sigma-\mu+(\tau-1)\kappa-\tau-h}\, r(H_{\omega-}, Q^*, (\omega-x)^{\tau+\mu+h-1})$$

$$\sim \sum_{h=0}^{\infty}{}'\, \gamma_h\, \omega^{\sigma-\mu+(\tau-1)\kappa-\tau-h}\, \zeta(1-\tau-\mu-h, \Theta)$$

where $\omega-\Theta$ is the largest integer $<\omega$. This gives the required asymptotic expansion for the sum $\sum\limits_{0<n<\omega} f(n)$.

4.7. Whittaker functions. Determine for large $|\omega|$ the asymptotic behavior of the sum

(4.7.1) $$\sum_{n=1}^{\infty} n^{m-k-\frac{1}{2}}\left(1+\frac{n}{u\omega}\right)^{m+k-\frac{1}{2}} e^{-n/\omega}$$

where

$$u \neq 0; \qquad -\tfrac{\pi}{2} < \arg\omega < \tfrac{\pi}{2}; \qquad -\pi+\varepsilon < \arg u + \arg\omega < \pi-\varepsilon$$

here k, m, u and $\varepsilon > 0$ denote fixed numbers. We shall prove that this sum is asymptotically equal to

(4.7.2) $$(m-k-\tfrac{1}{2})!\, e^{u/2}\, u^{-k}\, W_{km}(u)\omega^{m-k+\frac{1}{2}} + \sum_{n=0}^{\infty}{}'\, c_n\, \zeta(\tfrac{1}{2}+k-m-n)\, \omega^{-n}$$

where $W_{km}(u)$ is the Whittaker function and where

$$c_n = \sum_{h=0}^{n} \frac{(-)^{n-h}}{u^h\,(n-h)\,!} \binom{m+k-\frac{1}{2}}{h} \tag{4.7.3}$$

provided that $m-k-\frac{1}{2}$ is not a negative integer. If $m-k-\frac{1}{2}$ is a negative integer, then the asymptotic behavior of the sum under consideration can be determined by a passage to the limit.

The proof is simple. According to Theorem 2.2.2 the sum (4.7.1) is asymptotically equal to

$$\int\limits_{H_{0+}}^{\infty} x^{m-k-\frac{1}{2}} \left(1+\frac{x}{u\omega}\right)^{m+k-\frac{1}{2}} e^{-x/\omega}\,dx + r\,(H_{0+}, Q, f) \tag{4.7.4}$$

where f is the integrand of the integral and where Q is the periodic asymptotic neutrix with domain $\omega^\delta < x < 2\omega^\delta$; δ denotes a fixed positive number <1.

If $m-k-\frac{1}{2}$ is not a negative integer, then the integral occurring in (4.7.4) is equal to the first term in (4.7.2). Indeed, this is obvious if $\mathrm{Re}\,(m-k-\frac{1}{2}) > -1$, since then the integral is equal to the corresponding convergent integral taken from 0 to ∞, and follows in the case $\mathrm{Re}\,(m-k-\frac{1}{2}) \leqq -1$ from considerations of analyticity. Finally, in the interval $0 < x \leqq 2\omega^\delta$ we have

$$\begin{aligned} f\,(x) &= \sum_{h\geqq 0,\, g\geqq 0} \frac{(-)^g}{g\,!} \binom{m+k-\frac{1}{2}}{h} u^{-h}\,\omega^{-h-g}\,x^{m-k-\frac{1}{2}+h+g} \\ &= \sum_{n=0}^{\infty} c_n\, x^{m-k-\frac{1}{2}+n}\,\omega^{-n} \end{aligned}$$

by (4.7.3), so that

$$r\,(H_{0+}, Q, f) \sim \sum_{n=0}^{\infty}{}' c_n\,\zeta\,(\tfrac{1}{2}+k-m-n)\,\omega^{-n}.$$

This completes the proof.

4.8. Gamma integrals.

Theorem 4.8.1: Assume that α is real, b complex, $b-\alpha$ not $\geqq 0$, $\rho > 0$, σ complex, σ/ρ is not an integer $\leqq 0$; k is an

integer $\geqq 0$. Then the function

$$j(x) = (x-b)^{\sigma-1} \log^k (x-b)\, e^{-\omega^{-1}(x-b)^\rho} \qquad (x>\alpha)$$

has the property that the incomplete gamma integral

$$\int_\alpha^\infty j(x)\, dx$$

converges in each of the three cases treated in section 2.4, namely

$$(1) \qquad -\frac{\pi}{2} < \arg \omega < \frac{\pi}{2};$$

$$(2) \qquad -\frac{\pi}{2} \leqq \arg \omega \leqq \frac{\pi}{2}; \qquad \operatorname{Re} \sigma < \rho < 1;$$

$$(3) \qquad -\frac{\pi}{2} \leqq \arg \omega \leqq \frac{\pi}{2}; \qquad \rho = 1; \quad \operatorname{Re} \sigma < 0.$$

This incomplete gamma integral is equal to

$$(4.8.1) \qquad \frac{1}{\rho}\left(\frac{\partial}{\partial\sigma}\right)^k \left\{\left(\frac{\sigma}{\rho}-1\right)!\ \omega^{\sigma/\rho}\right\} - \sum_{m=0}^{\infty} \frac{(-)^m}{m!\,\omega^m}\, \psi(\alpha-b,\ \sigma+\rho m,\ k)$$

where ψ is defined in (3.1.5).

Proof: In the whole complex σ-plane, the points $-m\rho$ ($m=0, 1, \ldots$) excepted, the incomplete gamma integral and also the function (4.8.1) represent analytic functions of σ, so that in the proof I may assume that $\operatorname{Re} \sigma > 0$. Then the incomplete gamma integral can be written in the form

$$\int_\alpha^b j(x)\, dx + \int_0^\infty j(x+b)\, dx$$

$$= \sum_{m=0}^{\infty} \frac{(-)^m}{m!\,\omega^m} \int_\alpha^b (x-b)^{\sigma+\rho m-1} \log^k (x-b)\, dx + \int_0^\infty x^{\sigma-1} \log^k x\, e^{-\omega^{-1} x^\rho}\, dx$$

and this is equal to (4.8.1).

This result will be used to determine the asymptotic behavior of sums of the form

$$\sum_{n>a} g(n)\, e^{-\omega^{-1}(x-b)^{\rho}}$$

where $g(x)$ satisfies certain general conditions.

Let the real number a and the complex number b be fixed numbers such that b is not an integer $>a$. Assume that, for large positive x, the fixed function $g(x)$ is infinitely often differentiable, that $g^{(m)}(x)$ tends asymptotically to zero for $m \to \infty$, that $g(n)$ is defined for each integer $n > a$ and finally that $g(x)$ possesses at infinity the Hadamard expansion

$$g(x) \sim {\sum_{h=0}^{\infty}}' c_h\, (x-b)^{\sigma_h-1} \log^{k_h}(x-b) \tag{4.8.2}$$

with parameter b and exponents $\sigma_h - 1$ $(h=0, 1, \ldots)$. Our task is to determine for large $|\omega|$ the asymptotic behavior of

$$\sum_{n>a} f(n), \quad \text{where} \quad f(x) = g(x)\, e^{-\omega^{-1}(x-b)^{\rho}};$$

here ρ denotes a fixed positive number. We treat here three cases, similar to those treated above, namely

(1) $\quad -\dfrac{\pi}{2} + \varepsilon < \arg \omega < \dfrac{\pi}{2} - \varepsilon, \qquad$ where $\varepsilon > 0$ is fixed;

(2) $\quad -\dfrac{\pi}{2} \leqq \arg \omega \leqq \dfrac{\pi}{2}; \qquad \operatorname{Re} \sigma_h < \rho < 1 \; (h=0, 1, \ldots)$;

(3) $\quad -\dfrac{\pi}{2} \leqq \arg \omega \leqq \dfrac{\pi}{2}; \qquad \rho = 1; \quad \operatorname{Re} \sigma_h < 0 \; (h=0, 1, \ldots)$.

Theorem 4.8.2: *Under the conditions formulated above we have for large $|\omega|$, if none of the numbers σ_h/ρ $(h=0, 1, \ldots)$ is an integer $\leqq 0$,*

$$\sum_{n>a} f(n) \sim \frac{1}{\rho} {\sum_{h=0}^{\infty}}' c_h \left(\frac{\partial}{\partial \sigma_h}\right)^{k_h} \left\{(\rho^{-1}\sigma_h - 1)!\, \omega^{\sigma_h/\rho}\right\} + {\sum_{m=0}^{\infty}}' \frac{(-)^m}{m!\, \omega^m}\, \chi(-m\rho) \tag{4.8.3}$$

where in the whole complex s-plane, the points $\sigma_0, \sigma_1, \ldots$ excepted, $\chi(s)$ is the analytic continuation of the func-

tion represented for sufficiently large Re s by the series

$$\chi(s) = \sum_{n > a} (n-b)^{-s} g(n).$$

Proof: Let Q be the periodic asymptotic neutrix with domain $\alpha < \xi < 2\alpha$ with $\alpha = |\omega|^{\delta}$; here δ denotes a fixed positive number $< \rho^{-1}$. If $|\omega|$ is sufficiently large, then the fixed function $g(x)$ is infinitely often differentiable for $x \geqq \alpha$, since $\alpha \to \infty$ as $|\omega| \to \infty$. Theorem 2.2.2 yields

$$\sum_{n > \alpha} f(n) \sim \int_{\alpha}^{\infty} f(x)\, dx + r(\alpha, Q, f). \tag{4.8.4}$$

In the interval $\alpha \leqq x \leqq 2\alpha$ the asymptotic expansion

$$f(x) \sim {\sum_{h=0}^{\infty}}' \frac{(-)^h}{h!\,\omega^h} (x-b)^{h\rho} g(x)$$

holds uniformly in x, so that by Theorem 2.3.1

$$r(\alpha, Q, f) \sim {\sum_{m=0}^{\infty}}' \frac{(-)^m}{h!\,\omega^m}\, r(\alpha, Q, (x-b)^{m\rho} g(x)).$$

According to Theorem 2.1.1 the residues occurring on the right-hand side do not change their asymptotic behavior, if Q is replaced by the periodic asymptotic neutrix with domain $\alpha < x < \infty$, so that by Theorem 2.2.3 they do not change their asymptotic behavior if Q is replaced by the periodic neutrix P. Theorem 3.7.3 yields therefore

$$r(\alpha, Q, (x-b)^{m\rho} g(x)) \sim -\int_{\alpha}^{H\infty(b)} (x-b)^{m\rho} g(x)\, dx + \chi_1(-m\rho)$$

where in the whole complex s-plane, the points $\sigma_0, \sigma_1, \ldots$ excepted, $\chi_1(s)$ is the analytic continuation of the function represented for sufficiently large Re s by the series $\sum\limits_{n > \alpha} (n-b)^{-s} g(n)$.

The Hadamard expansion of $g(x)$ at infinity yields

$$-\int_{\alpha}^{H\infty(b)} (x-b)^{m\rho} g(x)\, dx \sim$$

$$\sim -\sum_{h=0}^{\infty}{}' c_h \int_{\alpha}^{H\infty(b)} (x-b)^{\sigma_h+m\rho-1} \log^{k_h}(x-b)\,dx$$

$$\sim \sum_{h=0}^{\infty}{}' c_h \psi(\alpha-b, \sigma_h+m\rho, k_h)$$

according to Theorem 3.1.1, so that

$$(4.8.5)\qquad r(\alpha, Q, (x-b)^{m\rho} g(x)) \sim \sum_{h=0}^{\infty}{}' c_h \psi(\alpha-\beta, \sigma_h+m\rho, k_h) + \chi_1(-m\rho).$$

Furthermore

$$\int_{\alpha}^{\infty} f(x)\,dx = \int_{\alpha}^{\infty} g(x)\, e^{-\omega^{-1}(x-b)^\rho}\,dx$$

$$\sim \sum_{h=0}^{\infty}{}' c_h \int_{\alpha}^{\infty} (x-b)^{\sigma_h-1} \log^{k_h}(x-b)\, e^{-\omega^{-1}(x-b)^\rho}\,dx$$

$$\sim u - \sum_{m=0}^{\infty}{}' \frac{(-)^m}{m!\,\omega^m} \sum_{h=0}^{\infty}{}' c_h \psi(\alpha-b, \sigma_h+m\rho, k_h)$$

according to the preceding theorem; here

$$u \sim \frac{1}{\rho} \sum_{h=0}^{\infty}{}' c_h \left(\frac{\partial}{\partial\sigma_h}\right)^{k_h} \{(\rho^{-1}\sigma_h - 1)!\,\omega^{\sigma_h/\rho}\}.$$

Using (4.8.4) and (4.8.5) we obtain therefore

$$\sum_{n>\alpha} f(n) \sim u + \sum_{m=0}^{\infty} \frac{(-)^m}{m!\,\omega^m} \chi_1(-m\rho).$$

Moreover

$$\sum_{a<n\leqq\alpha} f(n) = \sum_{a<n\leqq\alpha} g(n)\, e^{-\omega^{-1}(n-b)^\rho}$$

$$\sim \sum_{m=0}^{\infty}{}' \frac{(-)^m}{m!\,\omega^m} \sum_{a<n\leqq\alpha} g(n)\,(n-b)^{m\rho}.$$

Addition of these two results gives the required formula (4.8.3), where in the whole complex s-plane, the points $\sigma_0, \sigma_1, \ldots$ excepted, $\chi(s)$ is the

analytic continuation of the function represented for sufficiently large Re s by

$$\sum_{n>\alpha} (n-b)^{-s}\, g(n) + \sum_{a<n\leqq\alpha} (n-b)^{-s}\, g(n) = \sum_{n>a} (n-b)^{-s}\, g(n)$$

This completes the proof.

By choosing a real, $b>a$ such that the interval $a<x\leqq b$ does not contain an integer, we find for the function

$$f_k(x) = (x-b)^{\sigma-1} \log^k (x-b)\, e^{-\omega^{-1}(x-b)^\rho}$$

the following result:

Theorem 4.8.3. *If σ/ρ is not an integer $\leqq 0$, then*

$$(4.8.6)\qquad \sum_{n>b}^{\infty} f_0(n) \sim \frac{1}{\rho}\left(\frac{\sigma}{\rho}-1\right)!\, \omega^{\sigma/\rho} + \sum_{m=0}^{\infty} \frac{(-)^m}{m!\, \omega^m}\, \zeta(1-\sigma-m\rho, \vartheta)$$

where $b+\vartheta$ is the smallest integer $>b$. This formula may be differentiated term by term infinitely often with respect to σ; in other words, we have for each fixed integer $k\geqq 0$

$$\sum_{n>b} f_k(n) \sim \frac{1}{\rho}\left(\frac{\partial}{\partial\sigma}\right)^k \left((\rho^{-1}\sigma-1)!\, \omega^{\sigma/\rho}\right) + \sum_{m=0}^{\infty}{}' \frac{(-)^{m+k}}{m!\, \omega^m}\, \zeta^{(k)}(1-\sigma-m\rho, \vartheta)$$

where $\zeta^{(k)}(s, \vartheta)$ denotes the k^{th} derivative with respect to s of the function $\zeta(s, \vartheta)$.

Remark: In my paper: Neutrices, J. Soc. Indust. Appl. Math. **7**, No. 3, 1959, p. 253—279, I have chosen the particular case of this theorem with $b=0$ (hence $\vartheta=1$), $\sigma>1$ and $k=0$ as an illustration of the neutrix calculus.

Above we have excluded the values of σ for which σ/ρ is an integer $\leqq 0$, but, as we know, also for these values of σ the method yields the asymptotic behavior of the sum under consideration. For the sake of simplicity I restrict myself to the case $k=0$. If $\sigma=-l\rho$, where l is a fixed integer $\geqq 0$, then in (4.8.6) the first term on the right-hand side must be replaced by the constant term occurring in the Laurent expansion according to powers of s of the function

$$\frac{1}{\rho}\left(\frac{\sigma-s}{\rho}-1\right)!\ \omega^{(\sigma-s)/\rho}$$

$$=\frac{(-)^l}{l\,!\,\rho\,\omega^l}\left(-\frac{\rho}{s}+\log\omega-\Gamma_1+\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{l}\right)+o(1)$$

where Γ_1 is the constant of Euler and where $o(1)$ denotes a function of s which is analytic and equal to zero at $s=0$. Furthermore the term occurring on the right-hand side of (4.8.6) with $m=l$ must be replaced by $\frac{(-)^l}{l\,!\,\omega^l}\Gamma_\vartheta$, where Γ_ϑ is the constant defined by (4.1.4). In the case that $-\sigma/\rho$ is equal to an integer $l\geqq 0$ we find therefore

$$\sum_{n>b} f_0(n) \sim \frac{(-)^l}{l\,!\,\rho\,\omega^l}\left(\log\omega-\Gamma_1+\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{l}+\rho\,\Gamma_\vartheta\right)$$

$$+\sum_{\substack{h=0\\ h\neq l}}^{\infty}{}' \frac{(-)^h}{h\,!\,\omega^h}\,\zeta(1-\sigma-h\rho\,,\,\vartheta)\,.$$

CHAPTER 5. ASYMPTOTIC HADAMARD NEUTRICES

5.1. Definition of asymptotic Hadamard neutrices.

In the preceding sections we have determined under general conditions the asymptotic behavior of the difference between a sum and the corresponding integral. In the applications we have restricted ourselves almost always to cases in which it was easy to find the asymptotic behavior of the integral, but for the investigation of more complicated problems we need new asymptotic neutrices, namely asymptotic Hadamard neutrices.

Definition: Let a, α and β be real numbers depending on ω with $-\infty < a \leqq \alpha < \beta$, where $\log(1+\alpha-a)$ is asymptotically finite and where for each fixed positive ε

(5.1.1) $$\alpha - a = O(\omega^{\varepsilon}(\beta-\alpha))\,.$$

The asymptotic Hadamard neutrix A_{a+} with domain $\alpha<\xi<\beta$ is the class formed by the functions $\nu(\xi)$ $(\alpha<\xi<\beta)$ which, for each fixed real q, can be written in the form

$$\lambda(\xi)+O(\omega^{-q})$$

uniformly in ξ. Here $\lambda(\xi)$ is a linear combination of functions of the form $(\xi-a)^\sigma \log^k(\xi-a)$, where the couple $\sigma=0$, $k=0$ is not admitted; the number of terms of $\lambda(\xi)$, the complex exponents σ and the integers $k \geqq 0$ are independent of ω and ξ, but may depend on q; finally the coefficients occurring in the linear combination $\lambda(\xi)$ may depend on ω and q, but not on ξ.

If $\alpha<\beta\leqq a<\infty$, where $\log(1+a-\beta)$ is asymptotically finite and where for each fixed positive ε

$$a-\beta = O\,|\omega|^\varepsilon(\beta-\alpha) \tag{5.1.2}$$

then the asymptotic neutrix A_{a-} with domain $\alpha<\xi<\beta$ is defined in the same way, but now with the functions $(a-\xi)^\sigma \log^k(a-\xi)$ instead of $(\xi-a)^\sigma \log^k(\xi-a)$.

We shall show in Section 7.3 that A_{a+} and A_{a-} satisfy the asymptotic neutrix condition.

Let I be an arbitrary asymptotic neutrix with domain $\alpha<\xi<\beta$ and let b be a real number. Let $f(x)$ be integrable from ξ to b $(\alpha<\xi<\beta)$. If γ is independent of ξ and if for each fixed real q it is possible to find a function $\nu(\xi)$ negligible in I such that the order relation

$$\int_\xi^b f(x)\,dx = \gamma + \nu(\xi) + O\,|\omega|^{-q}$$

holds uniformly in ξ $(\alpha<\xi<\beta)$, then according to the convention made in Section 1.4 we write

$$\int_I^b f(x)\,dx \sim \gamma\,.$$

In view of this fact we call $f(x)$ integrable from I to b. Notice however that we have not defined the integral from I to b, but only its asymptotic behavior. Of course we have

$$\int_b^I f(x)\,dx \sim -\int_I^b f(x)\,dx\,.$$

If N is a neutrix or an asymptotic neutrix, then I say that a function $f(x)$

is integrable from I to N, if there exists a point b such that $f(x)$ is integrable from I to b and also from b to N; then we have

$$\int_I^N f(x)\,dx \sim -\int_N^I f(x)\,dx \sim \int_I^b f(x)\,dx + \int_b^N f(x)\,dx .$$

It is clear that the integral of $f(x)$ from I to N is independent of the choice of the point b.

Theorem 5.1.1: If a is real, σ is fixed and $p(t)$ denotes a fixed polynomial, then we have for each $b>a$

$$(5.1.3) \qquad \int_{A_{a+}}^{b} (x-a)^{\sigma-1} p(\log(x-a))\,dx \sim \int_H^b (x-a)^{\sigma-1} p(\log(x-a))\,dx$$

where $H = H_{a+}$ or $H_\infty(a)$, hence

$$\int_H^{A_{a+}} (x-a)^{\sigma-1} p(\log(x-a))\,dx \sim 0 .$$

We have for each $b<a$

$$(5.1.4) \qquad \int_b^{A_{a-}} (a-x)^{\sigma-1} p(\log(a-x))\,dx \sim \int_b^H (a-x)^{\sigma-1} p(\log(a-x))\,dx$$

where $H = H_{a-}$ or $H_{-\infty}(a)$, hence

$$\int_H^{A_{a-}} (a-x)^{\sigma-1} p(\log(x-a))\,dx \sim 0 .$$

Proof: It is sufficient to prove (5.1.3) with $p(t)=t^k$, since the proof for (5.1.4) runs in the same way. We have

$$\int_\xi^b (x-a)^{-1}\log^k(x-a)\,dx = (k+1)^{-1}\log^{k+1}(b-a) - (k+1)^{-1}\log^{k+1}(\xi-a)$$

where the last term is negligible in A_{a+}, H_{a+} and $H_\infty(a)$. This gives (5.1.3) in the case $\sigma=0$. If $\sigma\neq 0$, then

$$\int_{\xi}^{b} (x-a)^{\sigma-1} \log^k (x-a)\, dx = \psi(b-a, \sigma, k) - \psi(\xi-a, \sigma, k)$$

where ψ is defined in (3.1.5), so that the last term is negligible in A_{a+}, H_{a+} and $H_\infty(a)$. This completes the proof.

Definition : A function $f(x)$ is said to possess in an interval $\alpha<x<\beta$ an asymptotic Hadamard expansion according to powers of $x-a$ if it possesses for large $|\omega|$ uniformly in x $(\alpha<x<\beta)$ an expansion of the form

$$(5.1.5) \qquad f(x) \sim \sum_{h=0}^{\infty}{}' c_h (x-a)^{\sigma_h-1} p_h(\log(x-a))$$

with fixed exponents σ_h, with fixed polynomials $p_h(t)$ and with coefficients c_h which may depend on ω but not on x and which are for each fixed integer $h \geqq 0$ asymptotically finite.

A function $f(x)$ is said to possess in an interval $\alpha<x<\beta$ an asymptotic Hadamard expansion according to powers of $a-x$ if we use the expansion

$$(5.1.6) \qquad f(x) \sim \sum_{h=0}^{\infty}{}' c_h (a-x)^{\sigma_h-1} p_h(\log(a-x))$$

instead of (5.1.5).

5.2. Properties of asymptotic Hadamard neutrices.

Theorem 5.2.1 : Let A_{a+} be the asymptotic Hadamard neutrix with domain $\alpha<\xi<\beta$, where $\beta-\alpha$ is asymptotically finite. If a function $f(x)$ integrable from $\alpha+$ to β possesses in the interval $\alpha<x<\beta$, uniformly in x, the asymptotic expansion (5.1.5), then $f(x)$ is integrable from A_{a+} to β and

$$(5.2.1) \qquad \int_{A_{a+}}^{\beta} f(x)\,dx \sim \sum_{h=0}^{\infty}{}' c_h \int_{H_{a+}}^{\beta} (x-a)^{\sigma_h-1} p_h(\log(x-a))\,dx\,.$$

Remark : According to Theorem 3.1.1 we may replace the lower limit H_{a+} by $H_\infty(a)$. Notice that no condition is imposed on $f(x)$ outside the interval $\alpha<x<\beta$.

Proof: Without loss of generality we may assume that $p_h(t) = t^{k_h}$, where k_h is an integer $\geqq 0$, since the expansion occurring on the right-hand side of (5.1.5) can be replaced by a similar expansion in which each polynomial $p_h(\log(x-a))$ is replaced by a power of $\log(x-a)$ with an exponent which is an integer $\geqq 0$. Let us first show that the general term in the series occurring in (5.2.1) tends asymptotically to zero for $h \to \infty$; this implies the asymptotic convergence of the series itself according to the fundamental theorem of asymptotic series formulated in the remark occurring in Section 2.2.

By hypothesis the series occurring in (5.1.5) converges asymptotically, uniformly in x $(\alpha < x < \beta)$, so that

(5.2.2) $$c_h (x-a)^{\sigma_h - 1} \log^{k_h}(x-a)$$

tends asymptotically to zero for $h \to \infty$, uniformly in x $(\alpha < x < \beta)$, therefore also at $x = \beta$.

Now I distinguish three cases:

(1) Consider the possible values of ω for which $\beta - a \geqq 2$. According to the definition of ψ given in (3.1.5) we have

$$c_h \psi(\beta - a, \sigma_h, k_h) = O\, c_h (\beta - a)^{\sigma_h} \log^{k_h+1}(\beta - a).$$

Consequently $c_h \psi(\beta - a, \sigma_h, k_h)$ tends asymptotically to zero for $h \to \infty$, since by hypothesis $\beta - a$, therefore also $(\beta - a)\log(\beta - a)$ is asymptotically finite.

(2) Consider the possible values of ω for which $\beta - a \leqq \frac{1}{2}$. According to the definition of ψ given in (3.1.5) we have

$$c_h \psi(\beta - a, \sigma_h, k_h) = O\, c_h (\beta - a)^{\sigma_h} (1 + |\log(\beta - a)|^{k_h+1}).$$

Consequently $c_h \psi(\beta - a, \sigma_h, k_h)$ tends asymptotically to zero for $h \to \infty$, since

$$(\beta - \alpha)(1 + |\log(\beta - \alpha)|^{k_h+1})$$

is bounded for given k_h.

(3) Consider the possible values of ω for which $\frac{1}{2} < \beta - a < 2$. According to (5.1.1) we have

$$\tfrac{1}{2} < \beta - a = O\, |\omega|^{\varepsilon} (\beta - \alpha);$$

hence

$$(\beta - a)^{-1} = O\,|\omega|^{\varepsilon}$$

for each fixed positive number ε. Consequently the interval $\alpha < x < \beta$ contains certainly a point x such that

$$\log^{-1}(x - a) = O\,|\omega|^{\varepsilon}.$$

We choose $\varepsilon < 1/k_h$. Since (5.2.2) tends asymptotically to zero for $h \to \infty$, also c_h tends asymptotically to zero as $h \to \infty$. From the definition of ψ given in (3.1.5) it follows that

$$c_h\,\psi(\beta - a\,,\,\sigma_h\,,\,k_h) = O\,c_h$$

tends asymptotically to zero for $h \to \infty$.

In this way we have proved the asymptotic convergence of the series occurring in (5.2.1).

Now the last part of the proof. We have for $\alpha < \xi < \beta$

$$\int_{\xi}^{\beta} f(x)\,dx = \sum_{h=0}^{m-1} c_h \int_{\xi}^{\beta} (x-a)^{\sigma_h - 1} \log^{k_h}(x-a)\,dx + \int_{\xi}^{\beta} \rho_m(x)\,dx \tag{5.2.3}$$

where $\rho_m(x)$ tends asymptotically to zero for $m \to \infty$, uniformly in x $(\alpha < \xi < x < \beta)$. By hypothesis $\beta - \alpha$ is asymptotically finite so that the last term in (5.2.3) tends asymptotically to zero as $m \to \infty$. Consequently, if a fixed real number q is given, then we have for each sufficiently large fixed positive integer m

$$\int_{\xi}^{\beta} \rho_m(x)\,dx = O\,|\omega|^{-q} \tag{5.2.4}$$

uniformly in ξ $(\alpha < \xi < \beta)$. We can choose this fixed integer so large that it satisfies also the order relation

$$\begin{aligned} \sum_{h=0}^{\infty}{}' c_h \int_{H_{a+}}^{\beta} (x - a)^{\sigma_h - 1} \log^{k_h}(x - a)\,dx \\ = \sum_{h=0}^{m-1} c_h \int_{H_{a+}}^{\beta} (x - a)^{\sigma_h - 1} \log^{k_h}(x - a)\,dx + O\,|\omega|^{-q}. \end{aligned} \tag{5.2.5}$$

According to Theorem 5.1.1 the integral

$$\int_{A_{a+}}^{\beta} (x-a)^{\sigma_h-1} \log^{k_h} (x-a)\, dx$$

exists. This implies that

$$(5.2.6) \quad \int_{\xi}^{\beta} (x-a)^{\sigma_h-1} \log^{k_h}(x-a)\,dx = \int_{A_{a+}}^{\beta} (x-a)^{\sigma_h-1} \log^{k_h}(x-a)\,dx + \nu_h(\xi) + O\,|\omega|^{-q-p}$$

where $\nu_h(\xi)$ is negligible in A_{a+} and $c_h = O\,|\omega|^p$, so that (5.2.6) remains true if both sides are multiplied by c_h and $-q-p$ is replaced by $-q$. From the definition of the asymptotic neutrix A_{a+} it follows that also $c_h \nu_h(\xi)$ is negligible in A_{a+}. Consequently (5.2.6) yields

$$\sum_{h=0}^{m-1} c_h \int_{\xi}^{\beta} (x-a)^{\sigma_h-1} \log^{k_h} (x-a)\, dx$$

$$= \sum_{h=0}^{m-1} c_h \int_{A_{a+}}^{\beta} (x-a)^{\sigma_h-1} \log^{k_h} (x-a)\, dx + \nu(\xi) + O\,|\omega|^{-q},$$

where $\nu(\xi)$ is negligible in A_{a+}. According to Theorem 5.1.1 we may replace the lower limit A_{a+} by H_{a+}. Combining this result with (5.2.3), (5.2.4) and (5.2.5) we find therefore in the interval $\alpha < \xi < \beta$, uniformly in ξ,

$$\int_{\xi}^{\beta} f(x)\, dx = \sum_{h=0}^{\infty}{}' c_h \int_{H_{a+}}^{\beta} (x-a)^{\sigma_h-1} \log^{k_h} (x-a)\, dx + \nu(\xi) + O\,|\omega|^{-q}.$$

This holds for each fixed real q and yields therefore the required result.

Theorem 5.2.2: *If the conditions of the preceding theorem are satisfied with $\alpha = a$ and if there exists an integer $p \geqq 0$ such that $\operatorname{Re} \sigma_h > 0$ for each integer $h \geqq p$ and that*

$$f(x) - \sum_{h=0}^{p-1} c_h (x-a)^{\sigma_h-1} p_h(\log(x-a))$$

is integrable from a to β, then

(5.2.7) $$\int_{A_{a+}}^{\beta} f(x)\,dx \sim \int_{H_{a+}}^{\beta} f(x)\,dx \quad \text{hence} \quad \int_{H_{a+}}^{A_{a+}} f(x)\,dx \sim 0\,.$$

Remark : This theorem forms a bridge between the asymptotic Hadamard neutrices A_{a+} and the — in many respects simpler — neutrices H_{a+}. A similar remark holds for the Theorems 5.2.4, 5.3.1 and 5.3.2.

Proof : Since $\operatorname{Re}\sigma_h > 0$ for each $h \geqq p$, the function

$$\rho_m(x) = f(x) - \sum_{h=0}^{m-1} c_h (x-a)^{\sigma_h - 1} p_h(\log(x-a))$$

is, for each integer $m \geqq p$, integrable from a to β, so that

(5.2.8) $$\int_{H_{a+}}^{\beta} f(x)\,dx = \sum_{h=0}^{m-1} c_h \int_{H_{a+}}^{\beta} (x-a)^{\sigma_h-1} \log^{k_h}(x-a)\,dx + \int_{a}^{\beta} \rho_m(x)\,dx\,.$$

According to (5.1.5) the function $\rho_m(x)$ tends, for $m \to \infty$, asymptotically to zero uniformly in x $(a<x<\beta)$ and by hypothesis $\beta - a$ is asymptotically finite, so that the last term in (5.2.8) tends asymptotically to zero as $m \to \infty$. In this way we find for the left-hand side of (5.2.8) the same asymptotic expansion as in (5.2.1). This completes the proof.

In the same way we find the two following theorems.

Theorem 5.2.3 : Let A_{a-} be the asymptotic Hadamard neutrix with domain $\alpha<\xi<\beta$, where $\alpha<\beta\leqq a$ and where $\beta-\alpha$ is asymptotically finite. If a function $f(x)$ integrable from α to $\beta-$ possesses in the interval $\alpha<x<\beta$ the asymptotic Hadamard expansion (5.1.6), then $f(x)$ is integrable from α to A_{a-} and

$$\int_{\alpha}^{A_{a-}} f(x)\,dx \sim \sum_{h=0}^{\infty}{}' c_h \int_{\alpha}^{H_{a-}} (a-x)^{\sigma_h-1} p_h(\log(a-x))\,dx\,.$$

According to Theorem 3.1.2 this result remains true if the upper limit H_{a-} is replaced by $H_{-\infty}(a)$.

Theorem 5.2.4 : If the conditions of the preceding theorem are satisfied with $\beta = a$ and if there exists an

integer $p \geqq 0$ such that $\operatorname{Re} \sigma_h > 0$ for each integer $h \geqq p$ and that

$$f(x) - \sum_{h=0}^{p-1} c_h (a-x)^{\sigma_h - 1} p_h(\log(a-x))$$

is integrable from α to a, then

$$\int_{\alpha}^{A_{a-}} f(x)\,dx \sim \int_{\alpha}^{H_{a-}} f(x)\,dx \quad \text{hence} \quad \int_{A_{a-}}^{H_{a-}} f(x)\,dx \sim 0\,.$$

5.3. Asymptotic Hadamard neutrices with unbounded domains.

Theorem 5.3.1 : Assume that $a < \alpha$ and that $(\alpha - a)^{-1}$ is asymptotically finite. Consider the asymptotic Hadamard neutrix A_{a+} with domain $\alpha < \xi < \infty$. If $f(x)$ is integrable from α to $\infty-$ and if there exists a fixed positive number δ such that $(x-a)^{1+\delta} f(x)$ possesses in the interval $\alpha < x < \infty$ an asymptotic Hadamard expansion, according to powers of $x-a$, then $f(x)$ is integrable from α to A_{a+}.

If the said asymptotic Hadamard expansion has the form

$$(x-a)^{1+\delta} f(x) \sim \sum_{h=0}^{\infty}{}' c_h (x-a)^{\sigma_h + \delta} p_h(\log(x-a)) \tag{5.3.1}$$

then

$$\int_{\alpha}^{A_{a+}} f(x)\,dx \sim \sum_{h=0}^{\infty}{}' c_h \int_{\alpha}^{H_\infty(a)} (x-a)^{\sigma_h - 1} p_h(\log(x-a))\,dx\,. \tag{5.3.2}$$

Remark : According to Theorem 3.1.1 we may replace the upper limit $H_\infty(a)$ of the last integral by H_{a+}.

Proof : Exactly as in the proof of Theorem 5.2.1 we may assume here that $p_h(t) = t^{k_h}$, where k_h is an integer $\geqq 0$.

Let us first show that the general term in the series occurring in (5.3.2) tends asymptotically to zero for $h \to \infty$; this implies the asymptotic convergence of the series itself. It follows from (5.3.1) that

(5.3.3) $$c_h (x-a)^{\sigma_h+\delta} \log^{k_h} (x-a)$$

tends asymptotically to zero for $h \to \infty$ uniformly in x $(\alpha < x < \infty)$, therefore also at $x = \alpha$. Now I distinguish three cases:

(1) Consider the possible values of ω for which $\alpha - a \geqq 2$. Using the definition of ψ given in (3.1.5) we see that

$$\begin{aligned} c_h \psi(\alpha - a, \sigma_h, k_h) &= O\, c_h (\alpha - a)^{\sigma_h} \log^{k_h+1} (\alpha - a) \\ &= O\, c_h (\alpha - a)^{\sigma_h+\delta} \log^{k_h} (\alpha - a) \end{aligned}$$

tends asymptotically to zero for $h \to \infty$.

(2) Consider the possible values of ω for which $\alpha - a \leqq \frac{1}{2}$. Then

$$c_h \psi(\alpha - a, \sigma_h, k_h) = O\, c_h (\alpha - a)^{\sigma_h} (1 + |\log(\alpha - a)|^{k_h+1})$$

tends asymptotically to zero for $h \to \infty$, since by hypothesis $(\alpha - a)^{-1}$ and therefore also

$$(\alpha - a)^{-\delta} \log^{-k_h} (\alpha - a)\, (1 + |\log(\alpha - a)|^{k_h+1})$$

are asymptotically finite.

(3) Consider the possible values of ω for which $\frac{1}{2} < \alpha - a < 2$. Using the fact that (5.3.3) with $x = \alpha + 3$ tends asymptotically to zero for $h \to \infty$, we see that c_h and therefore also

$$c_h \psi(\alpha - a, \sigma_h, k_h) = O\, c_h$$

tend asymptotically to zero for $h \to \infty$.

Having shown in this way that the series occurring in (5.3.2) converges asymptotically, we write for $\xi > \alpha$

(5.3.4) $$\int_\alpha^\xi f(x)\, dx = \sum_{h=0}^{m-1} c_h \int_\alpha^\xi (x-a)^{\sigma_h - 1} \log^{k_h} (x-a)\, dx + \int_\alpha^\xi \rho_m(x)\, dx$$

where, according to (5.3.1), $(x-a)^{1+\delta} \rho_m(x)$ tends asymptotically to zero for $m \to \infty$, uniformly in x. Consequently for each fixed real q' the order relation

$$\rho_m(x) = O\,(x-a)^{-1-\delta} |\omega|^{-q'}$$

holds for sufficiently large m uniformly in x $(\alpha < x < \infty)$, so that the last term in (5.3.4) is

$$O\,|\omega|^{-q'} \int\limits_{\alpha}^{\infty} (x-a)^{-1-\delta}\,dx = O\,|\omega|^{-q'}(\alpha-a)^{-\delta}.$$

By hypothesis $(\alpha - a)^{-1}$ is asymptotically finite, so that the last term in (5.3.4) tends asymptotically to zero for $m \to \infty$. For each fixed real q it is therefore possible to find a fixed integer $m \geqq 0$ such that, uniformly in ξ $(\alpha < \xi < \infty)$

$$\int\limits_{\alpha}^{\xi} \rho_m(x)\,dx = O\,|\omega|^{-q}$$

and

$$\sum_{h=0}^{\infty}{}' c_h \int\limits_{\alpha}^{H_\infty(a)} (x-a)^{\sigma_h-1}\log^{k_h}(x-a)\,dx$$

$$= \sum_{h=0}^{m-1} c_h \int\limits_{\alpha}^{H_\infty(a)} (x-a)^{\sigma_h-1}\log^{k_h}(x-a)\,dx + O\,|\omega|^{-q}.$$

In view of the fact that the coefficients c_h are asymptotically finite we have

$$\sum_{h=0}^{m-1} c_h \int\limits_{\alpha}^{\xi} (x-a)^{\sigma_h-1}\log^{k_h}(x-a)\,dx$$

$$= \sum_{h=0}^{m-1} c_h \int\limits_{\alpha}^{A_{a+}} (x-a)^{\sigma_h-1}\log^{k_h}(x-a)\,dx + \nu(\xi)$$

where $\nu(\xi)$ is negligible in A_{a+}. The last integral does not change its asymptotic behavior according to Theorem 5.1.1 if the upper limit A_{a+} is replaced by $H_\infty(a)$. In this way we find for each fixed real q a function $\nu(\xi)$ negligible in A_{a+} such that the order relation

$$\int\limits_{\alpha}^{\xi} f(x)\,dx = \sum_{h=0}^{\infty}{}' c_h \int\limits_{\alpha}^{H_\infty(a)} (x-a)^{\sigma_h-1}\log^{k_h}(x-a)\,dx + \nu(\xi) + O\,|\omega|^{-q}$$

holds uniformly in ξ $(\alpha<\xi<\infty)$. This gives the required result.

Theorem 5.3.2: If the conditions of the preceding theorem are satisfied and if there exists an integer $p \geqq 0$ such that $\operatorname{Re}\sigma_h < 0$ for each integer $h \geqq p$ and that

$$f(x) - \sum_{h=0}^{p-1} c_h (x-a)^{\sigma_h - 1} p_h(\log(x-a))$$

is integrable from α to ∞, then

$$\int_{\alpha}^{A_{a+}} f(x)\,dx \sim \int_{\alpha}^{H_\infty(a)} f(x)\,dx \quad \text{hence} \quad \int_{A_{a+}}^{H_\infty(a)} f(x)\,dx \sim 0.$$

Proof: Since

$$f(x) - \sum_{h=0}^{m-1} c_h (x-a)^{\sigma_h - 1} p_h(\log(x-a))$$

is for each integer $m \geqq p$ integrable from α to ∞ it follows from (5.3.1) that

$$\int_{\alpha}^{H_\infty(a)} f(x)\,dx = \sum_{h=0}^{m-1} c_h \int_{\alpha}^{H_\infty(a)} (x-a)^{\sigma_h - 1} p_h(\log(x-a))\,dx + \int_{\alpha}^{\infty} \rho_m(x)\,dx \tag{5.3.5}$$

where, as we have seen in the proof of the preceding theorem, the last term tends asymptotically to zero for $m \to \infty$. In this way we find for the left-hand side of (5.3.5) the same asymptotic expansion as in (5.3.2). This completes the proof.

In the same way we find the two following theorems.

Theorem 5.3.3: Assume that $\beta < a$ and that $(a-\beta)^{-1}$ is asymptotically finite. Consider the asymptotic Hadamard neutrix A_{a-} with domain $-\infty < \xi < \beta$. If $f(x)$ is integrable from $-\infty+$ to β and if there exists a fixed positive number δ such that $(a-x)^{1+\delta} f(x)$ possesses in the interval $-\infty < x < \beta$ an asymptotic Hadamard expansion, then $f(x)$ is integrable from A_{a-} to β. If the said asymptotic

Hadamard expansion has the form

$$(a-x)^{1+\delta} f(x) \sim \sum_{h=0}^{\infty}{}' c_h (a-x)^{\sigma_h+\delta} p_h(\log(a-x))$$

then

$$\int_{A_{a-}}^{\beta} f(x)\,dx \sim \sum_{h=0}^{\infty}{}' c_h \int_{H_{-\infty}(a)}^{\beta} (a-x)^{\sigma_h-1} p_h(\log(a-x))\,dx .$$

Theorem 5.3.4: If the conditions of the preceding theorem are satisfied and if there exists an integer $p \geqq 0$ such that $\operatorname{Re}\sigma_h < 0$ for each integer $h \geqq p$ and that

$$f(x) - \sum_{h=0}^{p-1} c_h (a-x)^{\sigma_h-1} p_h(\log(a-x))$$

is integrable from $-\infty$ to β, then

$$\int_{A_{a-}}^{\beta} f(x)\,dx \sim \int_{H_{-\infty}(a)}^{\beta} f(x)\,dx \quad \text{hence} \quad \int_{H_{-\infty}(a)}^{A_{a-}} f(x)\,dx \sim 0 .$$

CHAPTER 6. APPLICATIONS

6.1. Beta integrals and hypergeometric functions.

Determine for large positive ω the asymptotic behavior of the sum

$$\sum_{0<n<\omega} f(n)$$

where

$$f(x) = x^{b-1}(\omega-x)^{c-b-1}(\omega-ux)^{-a}(v\omega^{\rho}+x)^{\tau} \qquad (x>0);$$

a, b, c, ρ, τ, u and v denote fixed numbers with

$$0<\rho<1; \qquad \operatorname{Re} c > \operatorname{Re} b > 0;$$

u is not a real number $\geqq 1$; v is not a real number $\leqq 0$.

According to the neutralized sum formula of Euler (Theorem 2.5.6) we have

$$(6.1.1) \qquad \sum_{0<n<\omega} f(n) \sim \int_{H_{0+}}^{H_{\omega-}} f(x)\,dx + r(H_{0+}, Q, f) + r(H_{\omega-}, Q^*, f);$$

Q and Q^* denote the periodic asymptotic neutrices respectively with domains $\alpha<x<\beta$ and $\alpha^*<x<\beta^*$, where

$$\alpha = \omega^{\delta}, \quad \beta = 2\omega^{\delta}; \quad \alpha^* = \omega - 2\omega^{\varepsilon}; \quad \beta^* = \omega - \omega^{\varepsilon};$$

δ and ε denote fixed positive numbers <1 with $\delta<\rho$.

Let us begin with the easiest part of our task, namely the determination of the asymptotic behavior of the two residues occurring in (6.1.1). In the interval $0<x\leqq 2\alpha$ we have

$$f(x) \sim \sum_{h=0}^{\infty}{}' c_h(\omega)\, x^{b-1+h}$$

hence

$$r(H_0+, Q, f) \sim \sum_{h=0}^{\infty}{}' c_h(\omega)\, \zeta(1-b-h)$$

if b is not an integer $\leqq 0$; if b is an integer $\leqq 0$ then we replace the term with $h=-b$ by $c_h(\omega)\,\Gamma_1$, where Γ_1 is the constant of Euler.

Putting $\omega - x = y$, we find in the interval $\alpha^* \leqq x < \omega$

$$f(x) \sim \sum_{h=0}^{\infty}{}' \gamma_h(\omega)\, y^{c-b-1+h}$$

so that, if $b-c$ is not an integer $\geqq 0$,

$$r(H_{\omega-}, Q^*, f) \sim \sum_{h=0}^{\infty}{}' \gamma_h(\omega)\, \zeta(1+b-c-h, \Theta);$$

here $\omega-\Theta$ is the largest integer $<\omega$. If $b-c$ is an integer $\geqq 0$, then we must replace the term with $h=b-c$ by $\gamma_h(\omega)\,\Gamma_\Theta$, where Γ_Θ is the constant defined in (4.1.4).

The only thing which still remains to be done is the determination of the asymptotic behavior of the integral occurring in (6.1.1). Write

$$(6.1.2) \qquad \int_{H_{0+}}^{H_{\omega-}} f(x)\,dx \sim \int_{H_{0+}}^{A_{0+}} f(x)\,dx + \int_{A_{0+}}^{H_{\omega-}} f(x)\,dx$$

where the domain of the asymptotic Hadamard neutrix A_{0+} is the interval $\omega^{\kappa}<x<2\omega^{\kappa}$, where $\varkappa$ denotes a fixed number with $\rho<\varkappa<1$.

This step is important. In order to determine the asymptotic behavior of the integral occurring on the left-hand side of (6.1.2) we partition the integration path into two parts by means of an asymptotic Hadamard neutrix and we determine separately the asymptotic behavior of the two integrals obtained in this way. This device, not only useful but in many problems even indispensable, can be explained as follows. The factor $(v\omega^{\rho}+x)^{\tau}$ occurring in the definition of $f(x)$ changes its behavior in the interval $0<x<\omega$. For small positive x the term $v\omega^{\rho}$ in the binomium $v\omega^{\rho}+x$ is preponderant, so that $(v\omega^{\rho}+x)^{\tau}$ possesses a binomium expansion according to powers of x with exponents 0, 1, 2, ... On the other hand, if $x<\omega$ lies in the neighborhood of ω, then because of $\rho<1$ the term x is preponderant, so that $(v\omega^{\rho}+x)^{\tau}$ possesses a binomium expansion according to powers of x with exponents τ, $\tau-1$, $\tau-2$, ... It is not possible to find for $(v\omega^{\rho}+x)^{\tau}$ a convenient expansion valid in the whole interval $0<x<\omega$. This is the reason why we partition the path into two parts and if in neutrix calculus we partition an integration path or an integration domain into two or more parts, then we prefer to do this by means of neutrices or asymptotic neutrices.

With purpose, I have chosen the positive number ρ less than 1 in this problem. The case $\rho>1$ is simpler, because then $(v\omega^{\rho}+x)^{\tau}$ possesses in the whole interval $0<x<\omega$ a binomium expansion according to ascending powers of x, so that then the partition of the integration path is not necessary with the consequence that we do not need an asymptotic Hadamard neutrix.

It is now our task to determine the asymptotic behavior of the two integrals occurring on the right-hand side of (6.1.2). Let us begin with the integral of $f(x)$ from H_{0+} to A_{0+}. In the corresponding interval $0<x<2\omega^{\kappa}$ the factor $(v\omega^{\rho}+x)^{\tau}$ changes its behavior because of $\rho<\varkappa<1$, as explained above. For this factor we have therefore no convenient expansion valid in the whole interval $0<x\leqq 2\omega^{\kappa}$ and we write

(6.1.3) $$f(x)\sim\sum_{h=0}^{l}{}' \lambda_h\,\omega^{c-a-b-h-1}\,x^{b+h-1}\,(v\omega^{\rho}+x)^{\tau}$$

where the coefficients λ_h are defined by

$$(6.1.4) \qquad (1-y)^{c-b-1}(1-uy)^{-a} = \sum_{h=0}^{\infty} \lambda_h y^h \quad \text{for} \quad 0<y<\min(1\,,\,|u|^{-1})\,.$$

Consequently λ_h is a polynomial in u of degree h. By means of the Theorems 3.2.1 and 5.2.1 it follows from (6.1.3) that

$$\int_{H_{0+}}^{A_{0+}} f(x)\,xd \sim \sum_{h=0}^{\infty}{}' \lambda_h\, \omega^{c-a-b-h-1} \int_{H_{0+}}^{A_{0+}} x^{b+h-1}(v\omega^\rho + x)^\tau\,dx\,.$$

According to Theorem 5.3.2 the last integral is asymptotically equal to the integral which we obtain by replacing the upper limit A_{0+} by $H_\infty(0)$. In order to determine the value of this new integral we use the fact that, for each fixed $\mu>0$, for each w with $-\pi<\arg w<\pi$ such that w and w^{-1} are asymptotically finite and for each fixed σ and each fixed τ such that neither σ/μ nor $-\sigma/\mu-\tau$ is an integer $\leqq 0$, the relation

$$(6.1.5) \qquad \int_{H_{0+}}^{H_\infty(0)} x^{\sigma-1}(w+x^\mu)^\tau\,dx = \frac{1}{\mu}\,\frac{(\frac{\sigma}{\mu}-1)\,!\,(-\frac{\sigma}{\mu}-\tau-1)\,!}{(-\tau-1)\,!}\,w^{(\sigma/\mu)+\tau}$$

holds. This formula is obvious for $\operatorname{Re}\sigma>0$ and $\operatorname{Re}(\sigma+\mu\tau)<0$, since then the integral is the convergent beta integral from 0 to ∞ with the value indicated by the right-hand side of (6.1.5); according to the Theorems 3.4.1 and 3.4.4 the left-hand side of (6.1.5) represents a function of σ and τ which is analytic in the whole complex σ-plane and the whole complex τ-plane, the points $\sigma = 0, -1, -2, \ldots$ and the points $\sigma+\tau = 0, 1, 2, \ldots$ excepted. This yields (6.1.5). Applying this formula with

$$\sigma = b+h\,; \quad w = v\omega^\rho \quad \text{and} \quad \mu = 1$$

we obtain

$$(6.1.6) \qquad \int_{H_{0+}}^{A_{0+}} f(x)\,dx$$

$$\sim \frac{1}{(-\tau-1)\,!} \sum_{h=0}^{\infty}{}' \lambda_h\, v^{b+\tau+h}\,(b+h-1)\,!\,(-b-\tau-h-1)\,!\;\omega^{c-a-1+\tau\rho-(1-\rho)(b+h)}$$

provided that b is not an integer $\leqq 0$ and $b+\tau$ is not an integer. In the exceptional cases the value of the integral is determined in the usual way.

For the evaluation of the integral of $f(x)$ from A_{0+} to $H_{\omega-}$ we use in the interval $\omega^\kappa \leqq x < \omega$ the binomium expansion

$$(v\omega^\rho + x)^\tau \sim \sum_{h=0}^{\infty}{}' \binom{\tau}{h} v^h \omega^{h\rho} x^{\tau-h}$$

hence by Theorem 5.2.1

$$\int_{A_{0+}}^{H_{\omega-}} f(x)\,dx \sim \sum_{h=0}^{\infty}{}' \binom{\tau}{h} v^h \omega^{h\rho} \int_{A_{0+}}^{H_{\omega-}} x^{b+\tau-h-1} (\omega - x)^{c-b-1} (\omega - ux)^{-a}\,dx\,.$$

According to Theorem 5.2.2 the last integral does not change its asymptotic behavior if the lower limit A_{0+} is replaced by H_{0+}. We have

$$(6.1.7) \qquad \int_{H_{0+}}^{H_{\omega-}} x^{b-1} (\omega - x)^{c-b-1} (\omega - ux)^{-a}\,dx$$

$$= \frac{(b-1)!\,(c-b-1)!}{(c-1)!} F(a\,,b\,,c\,;u)\,\omega^{c-a-1}$$

if neither b nor $c-b$ is an integer $\leqq 0$; here $F(a\,,b\,,c\,;u)$ is the hypergeometric function. This is obvious for $\operatorname{Re} c > \operatorname{Re} b > 0$, since then the left-hand side of (6.1.7) is the convergent hypergeometric integral from 0 to ω with the value indicated by the right-hand side of (6.1.7); according to the Theorems 3.4.1 and 3.4.2 the left-hand side of (6.1.7) represents a function of b and c which is analytic in the whole complex b-plane and the whole complex c-plane, the points $b = 0, -1, -2, \ldots$ and the points $c - b = 0, -1, -2, \ldots$ excepted; this yields (6.1.7) and therefore

$$(6.1.8) \qquad \int_{A_{0+}}^{H_{\omega-}} f(x)\,dx$$

$$\sim \sum_{h=0}^{\infty}{}' \binom{\tau}{h} v^h \frac{(b+\tau-h-1)!\,(c-b-1)!}{(c+\tau-h-1)!} F(a\,,b+\tau-h\,,c+\tau-h\,;u)\,\omega^{c-a-1+\tau-(1-\rho)h}$$

if $b+\tau$ is not an integer and $c-b$ is not an integer $\leqq 0$. In the exceptional cases the value of the integral is determined in the usual way.

The combination (6.1.6) and (6.1.8) gives by means of addition the asymptotic behavior of the integral of $f(x)$ from H_{0+} to $H_{\omega-}$. After this addition no term drops out, so that the partition of the integration path into two parts, one yielding the expansion indicated in (6.1.6) and the other yielding the expansion indicated in (6.1.8) is certainly appropriate for the determination of the asymptotic behavior of this integral.

6.2. On $\sum\limits_{n>0} (\omega^\kappa + \omega^\lambda n^\mu + n^\nu)^{-1}$.

Determine for large positive ω the asymptotic behavior of the sum

$$\sum_{n=1}^{\infty} f(n), \quad \text{where} \quad f(x) = (\omega^\kappa + \omega^\lambda x^\mu + x^\nu)^{-1}$$

where $\varkappa, \lambda, \mu, \nu$ denote fixed positive numbers. To insure the convergence of the series we assume that at least one of the two numbers μ and ν is >1. The method developed in this paper can be applied for each choice of the fixed parameters, but for the sake of simplicity I restrict myself to the case

$$\nu > \mu; \quad \varkappa > \lambda + \mu; \quad \mu\varkappa + \nu\lambda - \nu\varkappa > 0 \tag{6.2.1}$$

otherwise we have to distinguish several cases.

Formula (6.2.1) gives

$$0 < \frac{\varkappa - \lambda}{\mu} < \frac{\lambda}{\nu - \mu}. \tag{6.2.2}$$

Theorem 2.2.2 yields

$$\sum_{n=1}^{\infty} f(n) \sim \int_0^\infty f(x)\,dx + r(0, Q, f) \tag{6.2.3}$$

where Q is the periodic asymptotic neutrix with interval $\alpha < \xi < 2\alpha$; here $\alpha = \omega^\delta$ and δ denotes a fixed positive number $< \dfrac{\varkappa - \lambda}{\mu}$ and $< \dfrac{\varkappa}{\nu}$.

Let us first determine the asymptotic behavior of the residue occurring in (6.2.3). In the interval $0 \leqq x \leqq 2\alpha$ we have

$$f(x) \sim \sum_m \gamma_m \, \omega^{-m-\kappa}$$

where the sum is extended over all the numbers m which can be written in at least one way in the form

$$m = (\varkappa - \lambda)\, k + \varkappa l \tag{6.2.4}$$

where k and l are integers $\geqq 0$; furthermore

$$\gamma_m = \Sigma_1 (-)^{k+l} \binom{k+l}{l} x^{\mu k + \nu l}$$

where Σ_1 is extended over all the integers $k \geqq 0$ and $l \geqq 0$ with (6.2.4). Consequently

$$r(0, Q, f) \sim \sum_m{}' c_m \, \omega^{-m-\kappa}$$

where

$$c_m = \Sigma_1 (-)^{k+l} \binom{k+l}{l} \zeta(-\mu k - \nu l).$$

Finally we determine the asymptotic behavior of the integral of $f(x)$ from 0 to ∞. Here again we have an example of a function which changes its behavior in the integration interval. In view of this fact we introduce an asymptotic Hadamard neutrix A_{0+} with domain $\beta < \xi < 2\beta$, where $\beta = \omega^\rho$; here ρ denotes a fixed number with

$$\frac{\varkappa - \lambda}{\mu} < \rho < \frac{\lambda}{\nu - \mu}; \tag{6.2.5}$$

compare (6.2.2).

Put

$$\varepsilon = \min\left(\frac{\mu\varkappa + \nu\lambda - \nu\varkappa}{\mu},\ \lambda - (\nu - \mu)\,\rho\right)$$

so that $\varepsilon > 0$ according to (6.2.5) and the third of the inequalities (6.2.1). If

$$0 \leqq x \leqq \omega^{(\kappa-\lambda)/\mu}, \qquad \text{then} \qquad x^\nu \leqq \omega^{(\kappa-\lambda)\nu/\mu} \leqq \omega^{\kappa-\varepsilon}$$

and if

$$\omega^{(\kappa-\lambda)/\mu} < x \leqq 2\omega^{\rho}$$

then

$$x^{\nu} \leqq 2^{\nu-\mu}\, x^{\mu}\, \omega^{(\nu-\mu)\rho} \leqq 2^{\nu-\mu}\, x^{\mu}\, \omega^{\lambda-\varepsilon}$$

so that in the interval $0 \leqq x \leqq 2\beta$

$$x^{\nu} \leqq (\omega^{\kappa} + \omega^{\lambda}\, x^{\mu})\, 2^{\nu-\mu}\, \omega^{-\varepsilon}$$

hence

$$f(x) \sim \sum_{k=0}^{\infty}{}' (-)^h\, x^{\nu h}\, (\omega^{\kappa} + \omega^{\lambda}\, x^{\mu})^{-h-1}$$

therefore, according to Theorem 5.2.1,

$$\int_0^{A_{0+}} f(x)\, dx \sim \sum_{h=0}^{\infty}{}' (-)^h \int_0^{A_{0+}} x^{\nu h}\, (\omega^{\kappa} + \omega^{\lambda}\, x^{\mu})^{-h-1}\, dx\,.$$

According to Theorem 5.3.2 the last integral is asymptotically equal to the integral which we obtain by replacing the upper limit A_{0+} by $H_{\infty}(0)$ and this new integral is according to (6.1.5) equal to $a_h\, \omega^{-\sigma_h}$, where

$$\sigma_h = \mu^{-1}\left((\mu\kappa + \nu\lambda - \nu\kappa)\, h + \lambda - \kappa + \mu\kappa\right)$$

and

$$a_h = \mu^{-1}\left(\frac{\nu h}{\mu} + \frac{1}{\mu} - 1\right)!\,\left(-\frac{\nu-\mu}{\mu}\, h - \frac{1}{\mu}\right)!\,/\,h\,!$$

if $\dfrac{\nu-\mu}{\mu}\, h + \dfrac{1}{\mu}$ is not an integer >0. In this way we obtain

$$\int_0^{A_{0+}} f(x)\, dx \sim \sum_{h=0}^{\infty}{}' (-)^h\, a_h\, \omega^{-\sigma_h}\,.$$

The evaluation of the integral of $f(x)$ from A_{0+} to ∞ runs in the same way. For $x \geqq \beta > \omega^{\rho}$ we have

$$\omega^{\lambda-\kappa}\, x^{\mu} \geqq \omega^{\lambda-\kappa+\mu\rho}$$

where the last exponent is positive by (6.2.5), so that

$$f(x) \sim \sum_{h=0}^{\infty}{}' (-)^k\, \omega^{\kappa h}\, (\omega^{\lambda}\, x^{\mu} + x^{\nu})^{-h-1}$$

hence

$$\int_{A_{0+}}^{\infty} f(x)\,dx \sim \sum_{h=0}^{\infty}{}' (-)^h \omega^{\varkappa h} \int_{A_{0+}}^{\infty} x^{-(h+1)\mu} (\omega^{\lambda} + x^{\nu-\mu})^{-h-1}\,dx\,.$$

According to Theorem 5.3.2 the last integral is asymptotically equal to the integral which we obtain by replacing the lower limit A_{0+} by H_{0+} and this integral is by (6.1.5) equal to $b_h\,\omega^{-\tau_h}$, where

$$\tau_h = \frac{\nu(h+1)-1}{\nu-\mu}\,\lambda \quad \text{and} \quad b_h = \frac{1}{\nu-\mu}\left(\frac{-\mu h-\nu+1}{\nu-\mu}\right)!\,\left(\frac{\nu h+\mu-1}{\nu-\mu}\right)!\,/\,h\,!$$

if $\dfrac{\mu h+\nu-1}{\nu-\mu}$ is not an integer >0. In this way we obtain

$$\int_{A_{0+}}^{\infty} f(x)\,dx \sim \sum_{h=0}^{\infty}{}' (-)^h\, b_h\, \omega^{\varkappa h-\tau_h}$$

and therefore

$$\int_{0}^{\infty} f(x)\,dx \sim \sum_{h=0}^{\infty}{}' (-)^h\, a_h\, \omega^{-\sigma_h} + \sum_{h=0}^{\infty}{}' (-)^h\, b_h\, \omega^{\varkappa h-\tau_h}$$

unless the parameters assume the exceptional values mentioned above. The exceptional cases can be treated in the usual way.

6.3. On $\sum\limits_{n>0} (\omega^{\varkappa} + \omega^{\lambda} n^{\mu} + n^{\nu})^{-1} \sin \dfrac{n}{\omega}$.

In this section we determine the asymptotic behavior of the sum

$$\sum_{n=1}^{\infty}{}' f(n)$$

where

$$f(x) = (\omega^{\varkappa} + \omega^{\lambda} x^{\mu} + x^{\nu})^{-1} \sin \frac{x}{\omega} \qquad (x>0);$$

$\varkappa, \lambda, \mu, \nu$ denote fixed positive numbers. This problem is similar to that which we have treated in the preceding section, but here we do not need

an asymptotic Hadamard neutrix. I shall explain below why the introduction of such an asymptotic neutrix is superfluous.

To insure convergence I assume that at least one of the exponents μ and ν is >1. According to Theorem 2.2.2 we have

$$\sum_{n=1}^{\infty} f(n) \sim \int_0^{\infty} f(x)\,dx + r(0, Q, f) \tag{6.3.1}$$

where Q is the periodic asymptotic neutrix with domain $\alpha<\xi<2\alpha$; here $\alpha = \omega^{\delta}$ and δ denotes a fixed positive number $<\min\left(\frac{\varkappa-\lambda}{\mu}, \frac{\varkappa}{\nu}, 1\right)$.

In the interval $0 \leqq x \leqq 2\alpha$

$$f(x) \sim \sum_m{}' \gamma_m\, \omega^{-m-\kappa-1}$$

where the sum $\sum\limits_m$ is extended over all the numbers $m \geqq 0$ which can be written in at least one way in the form

$$m = 2h + (\varkappa-\lambda)\,k + \varkappa l \tag{6.3.2}$$

where h, k and l denote integers $\geqq 0$; furthermore

$$\gamma_m = \Sigma_1 \frac{(-)^{h+k+l}}{(2h+1)!} \binom{k+l}{l} x^{1+2h+\mu k+\nu l}$$

where Σ_1 is extended over all the integers $h \geqq 0$, $k \geqq 0$ and $l \geqq 0$ with (6.3.2). In this way we obtain

$$r(0, Q, f) \sim \sum_m{}' c_m\, \omega^{-m-\kappa-1}$$

where

$$c_m = \Sigma_1 \frac{(-)^{h+k+l}}{(2h+1)!} \binom{k+l}{l} \zeta(-1-2h-\mu k-\nu l).$$

Consequently the asymptotic behavior of the sum $\sum\limits_{n>0} f(n)$ is known as soon as the asymptotic behavior of the corresponding integral is known and the last problem is much simpler than that of the preceding section.

Indeed in the integral

$$(6.3.3) \qquad \int_0^{\infty} (\omega^{1-\kappa} + \omega^{\lambda} x^{\mu} + x^{\nu})^{-1} e^{ix/\omega}\, dx$$

$$= \omega \int_0^{\infty} (\omega^{\kappa} + \omega^{\lambda+\mu} t^{\mu} + \omega^{\nu} t^{\nu})^{-1} e^{it}\, dt$$

we can replace the integration path $(0\,,\,\infty)$ in the t-plane by the half-line $(0\,,\,\infty e^{ip})$, where p is a small positive number. For sufficiently small $|t|$ we have

$$\omega^{1-\kappa}(1 + \omega^{\lambda+\mu-\kappa} t^{\mu} + \omega^{\nu-\kappa} t^{\nu})^{-1} = \sum_{k\geq 0\,,\, l\geq 0} (-)^{k+l} \binom{k+l}{l} \omega^{-\sigma_{kl}} t^{\mu k+\nu l}$$

where

$$\sigma_{kl} = (\varkappa - \lambda - \mu)\, k + (\varkappa - \nu)\, l + \varkappa + 1\,,$$

so that the left-hand side of (6.3.3) is asymptotically equal to

$$\sum_{k\geqq 0\,,\, l\geqq 0}{}' (-)^{k+l} \binom{k+l}{l} \omega^{-\sigma_{kl}} \int_0^{\infty e^{ip}} t^{\mu k+\nu l} e^{it}\, dt$$

$$\sim \sum_{k\geq 0\,,\, l\geq 0}{}' (\ \ )^{k+l} \binom{k+l}{l} \exp\left[\mu k + \nu l + 1) \frac{\pi i}{2}\right] (\mu k + \nu l)!\, \omega^{-\sigma_{kl}}$$

hence

$$\int_0^{\infty} f(x)\, dx \sim \sum_{k\geqq 0\,,\, l\geqq 0}{}' (-)^{k+l} \binom{k+l}{l} (\mu k + \nu l)! \left(\sin\, (\mu k + \nu l + 1) \frac{\pi}{2}\right) \omega^{-\sigma_{kl}}\,.$$

In this example the asymptotic behavior of the integral is determined by the behavior of the integrand in the neighborhood of the origin and it does not make sense to partition the integration path into two or more parts, so that we do not need an asymptotic Hadamard neutrix.

6.4. Determine for large positive ω the asymptotic behavior of

$$\sum_{0<n<\omega} \left(\sin \frac{\pi\,(n^2+1)}{\omega^2+1}\right)^{\lambda}$$

where λ denotes a fixed complex number. Put

$$f(x) = \left(\sin \frac{\pi(x^2+1)}{\omega^2+1}\right)^{\lambda}.$$

The sum under consideration is according to the neutralized sum formula of Euler (Theorem 2.5.6) asymptotically equal to

$$\int_0^{H_{\omega-}} f(x)\,dx + r(0, Q, f) + r(H_{\omega-}, Q^*, f) \tag{6.4.1}$$

where Q and Q^* are the periodic asymptotic neutrices respectively with the domains $\alpha < x < 2\alpha$ and $\omega - 2\alpha < x < \omega - \alpha$; here $\alpha = \omega^{\delta}$ and δ is a fixed positive number <1. Put

$$(\sin \pi y)^{\lambda} = \sum_{h=0}^{\infty} c_h\, y^{2h+\lambda}. \tag{6.4.2}$$

In the interval $0 < x \leqq 2\alpha$ we have

$$\left(\sin \frac{\pi(x^2+1)}{\omega^2+1}\right)^{\lambda} \sim \sum_{h=0}^{\infty}{}' c_h (\omega^2+1)^{-2h-\lambda} (x^2+1)^{2h+\lambda}$$

so that

$$r(0, Q, f) \sim \sum_{h=0}^{\infty}{}' c_h (\omega^2+1)^{-2h-\lambda}\, r(0, Q, (x^2+1)^{2h+\lambda}). \tag{6.4.3}$$

The last residue is asymptotically equal to $r(0, P, (x^2+1)^{2h+\lambda})$; $r(0, P, (x^2+1)^{-s/2})$ is according to assertion (1) of Theorem 3.7.1 the entire function of s which in the half plane $\operatorname{Re} s > 1$ is represented by

$$\sum_{n=1}^{\infty} (n^2+1)^{-s/2} - \int_0^{\infty} (x^2+1)^{-s/2}\,dx.$$

If we put $x = \omega - y$, then we have for $\omega - 2\alpha \leqq x < \omega$

$$\left(\sin \frac{\pi(x^2+1)}{\omega^2+1}\right)^{\lambda} = \left(\sin \frac{\pi(2\omega y - y^2)}{\omega^2+1}\right)^{\lambda} \sim \sum_{h=0}^{\infty}{}' c_h \left(\frac{2\omega y - y^2}{\omega^2+1}\right)^{2h+\lambda}$$

$$\sim \sum_{h \geqq 0,\, k \geqq 0}{}' (-)^k c_h \binom{2h+\lambda}{k} (2\omega)^{-k} \left(\frac{2\omega}{\omega^2+1}\right)^{2h+\lambda} y^{2h+k+\lambda}$$

so that

$$r(H_{\omega-}, Q^*, f) \sim \sum_{h \geqq 0,\, k \geqq 0}{}' (-)^k c_h \binom{2h+\lambda}{k} (2\omega)^{-k} \left(\frac{2\omega}{\omega^2+1}\right)^{2h+\lambda}$$
$$\times\, r(H_{\omega-}, Q^*, (\omega - x)^{2h+k+\lambda}).$$

The last residue is asymptotically equal to

$$r(H_{\omega-}, -P, (\omega - x)^{2h+k+\lambda}) = \zeta(-2h - k - \lambda, \vartheta) \tag{6.4.4}$$

if $2h + k + \lambda \neq -1$; here $\omega - \vartheta$ is the largest integer $< \omega$. If $2h + k + \lambda = -1$, then we must replace the right-hand side of (6.4.4) by the constant Γ_ϑ defined in (4.1.4). This gives the asymptotic behavior of the two residues occurring in (6.4.1), so that the only thing we have still to do is the determination of the asymptotic behavior of the integral occurring in (6.4.1).

The substitution $(x^2+1)/(\omega^2+1) = y$ leads to

$$\int_0^{H_{\omega-}} \left(\sin \frac{\pi(x^2+1)}{\omega^2+1}\right)^\lambda dx \tag{6.4.5}$$
$$= \frac{1}{2}(\omega^2+1)^{1/2} \int_{(\omega^2+1)^{-1}}^{H_{1-}} (\sin \pi y)^\lambda (y - (\omega^2+1)^{-1})^{-1/2}\, dy.$$

This formula is obvious if $\operatorname{Re}\lambda > -1$, since then the two integrals are convergent integrals with the required property. Considerations of analyticity show the validity for each λ which is not a negative integer, since according to Theorem 3.4.2 the two integrals are analytic functions of λ in the whole complex λ-plane, the points $-1, -2, \ldots$ excepted.

In order to determine the asymptotic behavior of the integral occurring on the right-hand side of (6.4.5), I put for the sake of simplicity, $(\omega^2+1)^{-1} = w$ and I introduce the asymptotic Hadamard neutrix A_{0+} with domain $\beta < x < 2\beta$, where $\beta = w^\delta$. In the interval $0 < y \leqq 2\beta$ we can use (6.4.2), so that

$$\int_w^{A_{0+}} (\sin \pi y)^\lambda (y - w)^{-1/2}\, dy \sim \sum_{h=0}^{\infty}{}' c_h \int_w^{A_{0+}} y^{2h+\lambda} (y - w)^{-1/2}\, dy.$$

The last integral is asymptotically equal to

$$\text{(6.4.6)} \qquad \int_{w}^{H_\infty(0)} y^{2h+\lambda}(y-w)^{-1/2}\,dy = \int_{0}^{H_\infty(0)} (x+w)^{2h+\lambda}\,x^{-1/2}\,dx$$

if $2h+\lambda \neq -\frac{1}{2}, -\frac{3}{2}, \ldots$. This formula is obvious for $\operatorname{Re}(2h+\lambda) < -\frac{1}{2}$, since then the two integrals are convergent integrals with upper limits ∞ and both integrals represent, according to Theorem 3.4.3, analytic functions of λ in the complex λ-plane, the points $\lambda+2h = -\frac{1}{2}, -\frac{3}{2}, \ldots$ excepted. This yields (6.4.6), where the right-hand side is, according to (6.1.5), equal to

$$\pi^{1/2}\,\frac{(-2h-\lambda-\frac{3}{2})\,!}{(-2h-\lambda-1)\,!}\,w^{2h+\lambda+1/2}$$

if $2h+\lambda \neq -\frac{1}{2}, -\frac{3}{2}, \ldots$. In this way we obtain

$$\text{(6.4.7)} \qquad \int_{w}^{A_{0+}} (\sin \pi y)^{\lambda}\,(y-w)^{-1/2}\,dy$$

$$\sim \pi^{1/2} \sum_{h=0}^{\infty}{}' c_h\,\frac{(-2h-\lambda-\frac{3}{2})\,!}{(-2h-\lambda-1)\,!}\,w^{2h+\lambda+1/2}$$

if 2λ is not an odd integer.

To determine the asymptotic behavior of the contribution to the right-hand side of (6.4.5) by the path between A_{0+} and H_{1-} we write in the interval $\beta \leqq y < 1$

$$(y-w)^{-1/2} \sim \sum_{h=0}^{\infty}{}'(-)^h \binom{-1/2}{h} w^h\,y^{-h-1/2}$$

so that

$$\int_{A_{0+}}^{H_{1-}} (\sin \pi y)^{\lambda}\,(y-w)^{-1/2}\,dy$$

$$\sim \sum_{h=0}^{\infty}{}'(-)^h \binom{-1/2}{h} w^h \int_{A_{0+}}^{H_{1-}} (\sin \pi y)^{\lambda}\,y^{-h-1/2}\,dy\,.$$

According to Theorem 5.1.1 the last integral does not change its asymptotic behavior if the lower limit A_{0+} is replaced by H_{0+}. Applying (6.4.5) and (6.4.7) we find therefore

$$\int_{0}^{H_{\omega-}} \left(\sin \frac{\pi (x^2 - 1)}{\omega^2 + 1}\right)^{\lambda} dx \tag{6.4.8}$$

$$\sim \frac{1}{2}\sqrt{\pi} \sum_{h=0}^{\infty}{}' c_h \frac{(-2h - \lambda - \frac{3}{2})\,!}{(-2h - \lambda - 1)\,!} (\omega^2 + 1)^{-2h-\lambda}$$

$$+ \frac{1}{2} \sum_{h=0}^{\infty}{}' (-)^h \binom{-\frac{1}{2}}{h} (\omega^2 + 1)^{\frac{1}{2}-h} \int_{H_{0+}}^{H_{1-}} (\sin \pi y)^{\lambda} y^{-h-\frac{1}{2}} dy$$

if 2λ is not an odd integer. Notice that in the last asymptotic series the coefficients represented by integrals are fixed numbers and that these coefficients can be evaluated with any desired degree of accuracy.

If 2λ is an odd integer, then the asymptotic value of the integral occurring on the left-hand side of (6.4.8) is determined in the usual way.

6.5. On a problem similar to that treated in Section 4.3.

In Section 4.3 we have determined the asymptotic behavior of the sum

$$\sum_{0<n<2\omega} (n(\omega - n)(2\omega - n))^{1/3}.$$

In this section we shall determine for large positive ω the asymptotic behavior of the sum

$$\sum_{0<n<2\omega} f(n), \quad \text{where} \quad f(x) = (x(x^2 + 1)(\omega - x)(2\omega - x))^{1/3}.$$

According to the neutralized sum formula of Euler (Theorem 2.5.6), this sum is asymptotically equal to the integral

$$\int_{0}^{2\omega} f(x)\, dx$$

augmented with the corresponding residues at $0+$, $\omega-$, $\omega+$ and $2\omega-$. The asymptotic behavior of these residues is determined in the same way as in

Section 4.3, but the integral offers here more difficulties. This integral is equal to

$$(6.5.1)\qquad 2^{1/3}\,\omega^{8/3}\int_0^2 y^{1/3}(1-y)^{1/3}\left(1-\frac{y}{2}\right)^{1/3}\left(y^2+\frac{1}{\omega^2}\right)^{1/3}dy$$

where we have again to deal with the phenomenon that the factor $\left(y^2+\frac{1}{\omega^2}\right)^{1/3}$ changes its behavior in the interval $0<y<2$; for $y>\omega^{-1}$ the term y^2 is preponderant in the binomium $y^2+\frac{1}{\omega^2}$, but for $0<y<\omega^{-1}$ the other term is preponderant. In view of this fact we introduce the asymptotic Hadamard neutrix A_{0+} with domain $\alpha<\xi<2\alpha$; here $\alpha=\omega^{-\delta}$ and δ denotes a fixed positive number <1. If we define the coefficients c_h by the expansion

$$(1-y)^{1/3}\left(1-\frac{y}{2}\right)^{1/3}=\sum_{h=0}^{\infty} c_h y^h \quad \text{for} \quad |y|<1$$

then

$$\int_0^{A_{0+}} y^{1/3}(1-y)^{1/3}\left(1-\frac{y}{2}\right)^{1/3}\left(y^2+\frac{1}{\omega^2}\right)^{1/3}dy$$

$$\sim\sum_{h=0}^{\infty}{}' c_h\int_0^{A_{0+}} y^{h+1/3}\left(y^2+\frac{1}{\omega^2}\right)^{1/3}dy\,.$$

The last integral does not change its asymptotic behavior if A_{0+} is replaced by $H_\infty(0)$ and is therefore according to (6.1.5) asymptotically equal to

$$\frac{(\frac{1}{2}h-\frac{1}{3})!\,(-\frac{1}{2}h-2)!}{2\,(-\frac{4}{3})!}\,\omega^{-h-2}$$

so that

$$(6.5.2)\qquad \int_0^{A_{0+}} y^{1/3}(1-y)^{1/3}\left(1-\frac{y}{2}\right)^{1/3}\left(y^2+\frac{1}{\omega^2}\right)^{1/3}dy$$

$$\sim\frac{1}{2\,(-\frac{4}{3})!}\sum_{h=0}^{\infty}{}' c_h\left(\frac{1}{2}h-\frac{1}{3}\right)!\left(-\frac{1}{2}h-2\right)!\,\omega^{-h-2}\,.$$

Furthermore

$$\int_{A_{0+}}^{2} y^{1/3}(1-y)^{1/3}\left(1-\frac{y}{2}\right)^{1/3}\left(y^2+\frac{1}{\omega^2}\right)^{1/3} dy$$

$$\sim \sum_{h=0}^{\infty}{}' \binom{\frac{1}{3}}{h} \omega^{-2h} \int_{A_{0+}}^{2} y^{1-2h}(1-y)^{1/3}\left(1-\frac{y}{2}\right)^{1/3} dy$$

where according to Theorem 5.2.2 the last integral does not change its asymptotic behavior if the lower limit is replaced by H_{0+}. In this way we find

$$\int_{A_{0+}}^{2} y^{1/3}(1-y)^{1/3}\left(1-\frac{y}{2}\right)^{1/3}\left(y^2+\frac{1}{\omega^2}\right)^{1/3} dy$$

$$\sim \sum_{h=0}^{\infty}{}' \binom{\frac{1}{3}}{h} \omega^{-2h} \int_{H_{0+}}^{2} y^{1-2h}(1-y)^{1/3}\left(1-\frac{y}{2}\right)^{1/3} dy .$$

This, in conjunction with (6.5.2), gives the asymptotic behavior of the integral occurring in (6.5.1).

CHAPTER 7. FINALE

7.1. The periodic neutrix and the periodic asymptotic neutrix.

Definition: A neutrix N with domain $\alpha<\xi<\infty$ is called smooth if each of its functions can be written as a finite sum of smooth functions and if moreover each function of ξ $(\alpha<\xi<\infty)$ which tends to zero for $\xi \to \infty$ is negligible in N.

In Section 1.5 we have introduced smooth functions and moreover a certain class P of functions. In this section we shall show that P satisfies the neutrix condition. We shall even prove more, namely that P is compatible with each smooth neutrix N with the same domain $\alpha<\xi<\infty$. Two neutrices M and N with the same domain N' are called compatible if the functions $\mu(\xi)+\nu(\xi)$, where $\mu(\xi)$ denotes an arbitrary function negligible in M and where $\nu(\xi)$ denotes an arbitrary function negligible in N, form

a neutrix with domain N'. This neutrix is called the sum $M+N$ of the two compatible neutrices.

The statement that P and an arbitrary smooth function N are compatible, means therefore: if a function $\pi(\xi)$ negligible in P and a function $\nu(\xi)$ negligible in the smooth neutrix N have the property that

$$\pi(\xi)+\nu(\xi) \quad \gamma \qquad (\alpha<\xi<\infty)$$

where γ is independent of ξ, then $\gamma=0$. Choosing $\nu(\xi)=0$ we find in particular that P satisfies the neutrix condition.

In Section 2.2 we have introduced functions which are asymptotically smooth in an interval $\alpha<x<\beta$. An asymptotically smooth neutrix N with domain $\alpha<\xi<\beta$ is a neutrix such that, for each fixed real q, each function $\nu(\xi)$ negligible in N can be written in the form $s(\xi)+O\,|\omega|^{-q}$ uniformly in ξ $(\alpha<\xi<\beta)$, where $s(\xi)$ is the sum of a finite number of functions which are asymptotically smooth in $\alpha<\xi<\beta$; the number of the terms may depend on q but not on ω and ξ.

We shall prove that the classes Q introduced in Section 2.2 satisfy the asymptotic neutrix condition. We shall prove more, namely that Q is compatible with each asymptotically smooth neutrix N with the same domain $\alpha<\xi<\beta$. This means that the functions $\pi(\xi)+\nu(\xi)$, where $\pi(\xi)$ is an arbitrary function negligible in Q and where $\nu(\xi)$ is an arbitrary function negligible in N, form an asymptotic neutrix with domain $\alpha<\xi<\beta$.

P r o o f t h a t Q i s c o m p a t i b l e w i t h e a c h a s y m p t o t i c a l l y s m o o t h n e u t r i x N w i t h t h e s a m e d o m a i n $\alpha<\xi<\beta$. We must prove for each number γ which is independent of ξ but may depend on ω: if for each fixed real q it is possible to find a function $\pi(\xi)$ negligible in Q and a function $\nu(\xi)$ negligible in the smooth neutrix N such that the order relation

$$\pi(\xi)+\nu(\xi)=\gamma+O\,|\omega|^{-q} \tag{7.1.1}$$

holds uniformly in ξ $(\alpha<\xi<\beta)$, then $\gamma\sim 0$.

By hypothesis $\pi(\xi)$ can be written in the form (2.2.1); $\nu(\xi)$ can be written as a sum $\sum\limits_{h=n}^{n+m-1} s_h(\xi)+O\,|\omega|^{-q}$, where $s_h(\xi)$ $(h=n, n+1, \dots, n+m-1)$ is asymptotically smooth in $\alpha<\xi<\beta$; the integers $m\geqq 0$ and $n\geqq 0$ may depend on q but are independent of ω and ξ. Formula (7.1.1) can there-

fore be written as

$$(7.1.2) \qquad \gamma = \sum_{h=0}^{m+n-1} s_h(\xi)\, p_h(\xi) + O\,|\omega|^{-q}$$

where $p_h(\xi) = 1$ $(h = n\,, n+1\,, \ldots, n+m-1)$. Consequently the functions $p_0(\xi), \ldots, p_{m+n-1}(\xi)$ are periodic functions of ξ with period 1.

Since the functions $s_0(\xi), \ldots, s_{m+n-1}(\xi)$ are asymptotically smooth in the interval $\alpha<\xi<\beta$ it is possible to find fixed integers $\rho_0, \ldots, \rho_{m+n-1}$ each $\geqq 0$ such that the order relation

$$(7.1.3) \qquad s_h^{(\rho_h)}(\xi) = O\,|\omega|^{-q}$$

holds uniformly in ξ $(\alpha<\xi<\beta)$. Put

$$\rho = 1 + \max_{0 \leqq h < m+n} \rho_h\,.$$

According to the definition of Q given in Section 2.2 the difference $\beta-\alpha$ tends to infinity for $|\omega| \to \infty$, so that $\beta-\alpha > 2\rho$ for sufficiently large $|\omega|$. If λ is an arbitrary number with $\alpha < \lambda \leqq \frac{1}{2}(\alpha+\beta)$, then (7.1.3) holds certainly uniformly in ξ and λ in the interval $\lambda \leqq \xi \leqq \lambda + \rho$. In this interval we have therefore for $h = 0, 1, \ldots, m+n-1$

$$s_h(\xi) = \sum_{l=0}^{\rho_h-1} b_{hl}(\xi-\lambda)^l + O\,|\omega|^{-q}$$

where

$$(7.1.4) \qquad b_{hl} = \frac{1}{l\,!}\, s_h^{(l)}(\lambda)\,.$$

Using the fact that according to the definition of Q given in Section 2.2 the functions $p_h(\xi)$ are bounded functions of ω and ξ, we can therefore write (7.1.2) as

$$\gamma = v(\xi\,, \lambda) + O\,|\omega|^{-q}$$

where

$$v(\xi\,, \lambda) = \sum_{l=0}^{\rho-2} \chi_l(\xi)\,(\xi-\lambda)^l \quad \text{and} \quad \chi_l(\xi) = \sum_{\substack{0 \leqq h < m+n \\ \rho_h > l}} b_{hl}\, p_h(\xi)\,.$$

Applying this result with $\xi = \lambda + \mu + u$, where $0 \leq \mu \leq 1$ and $u = 0, 1, \ldots, \rho - 2$, we obtain

(7.1.5) $$\gamma = v(\lambda + \mu + u, \lambda) + O|\omega|^{-q}$$

where

$$v(\lambda + \mu + u, \lambda) = \sum_{l=0}^{\rho-2} \chi_l(\lambda + \mu + u)(\mu + u)^l.$$

Since the functions $p_0(x), \ldots, p_{m+n-1}(x)$ are by hypothesis periodic functions of x with period 1 it follows from the definition of the functions χ that also these functions $\chi_0(x), \ldots, \chi_{\rho-2}(x)$ have the period 1. Consequently

(7.1.6) $$v(\lambda + \mu + u, \lambda) = \sum_{l=0}^{\rho-2} \chi_l(\lambda + \mu)(\mu + u)^l$$

for $u = 0, 1, \ldots, \rho - 2$. The determinant Δ of Vandermonde

$$\Delta = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \mu & \mu + 1 & \cdots & \mu + \rho - 2 \\ \cdot & \cdot & \cdot & \cdot \\ \mu^{\rho-2} & (\mu+1)^{\rho-2} & \cdots & (\mu + \rho - 2)^{\rho-2} \end{vmatrix}$$

depends only on ρ and is $\neq 0$. If we multiply both sides of (7.1.6) with the minor Δ_u in Δ of the element in the first row and $(u+1)^{\text{st}}$ column and if we add the $\rho - 1$ relations obtained in this way, then we find

$$\sum_{u=0}^{\rho-2} v(\lambda + \mu + u, \lambda)\,\Delta_u = \chi_0(\lambda + \mu)\,\Delta,$$

so that (7.1.5) yields

$$\gamma\Delta = \sum_{u=0}^{\rho-2} \gamma\Delta_u = \chi_0(\lambda + \mu)\,\Delta + O|\omega|^{-q}$$

hence

$$\gamma = \chi_0(\lambda + \mu) + O|\omega|^{-q}.$$

Using (7.1.4) and the definition of $\chi_0(\xi)$ we find therefore

$$(7.1.7) \qquad \gamma = \sum_{\substack{0 \leq h < m+n \\ \rho_h \neq 0}} s_h(\lambda)\, p_h(\lambda + \mu) + O\,|\omega|^{-q}.$$

For the possible integers h with $0 \leq h < m + n$ and $\rho_h = 0$ we have according to (7.1.3)

$$s_h(\lambda) = O\,|\omega|^{-q}, \quad \text{hence} \quad s_h(\lambda)\, p_h(\lambda + \mu) = O\,|\omega|^{-q}$$

since $p_h(\lambda + \mu)$ is bounded according to the definition of Q given in Section 2.2. Consequently (7.1.7), remains true if the left-hand side is augmented with the terms corresponding to these values of h; in other words

$$(7.1.8) \qquad \gamma = \sum_{h=0}^{m+n-1} s_h(\lambda)\, p_h(\lambda + \mu) + O\,|\omega|^{-q}.$$

This formula holds uniformly in λ and μ.

According to formula (2.2.2) we have

$$\int_0^1 p_h(\lambda + \mu)\, d\mu = 0 \qquad (h = 0, 1, \ldots, n-1)$$

and moreover

$$\int_0^1 p_h(\lambda + \mu)\, d\mu = 1 \qquad (h = n\,, n+1, \ldots, n+m-1)$$

since $p_h(x) = 1$ for these values of h. Integrating both sides of (7.1.8) with respect to μ from 0 to 1 we find therefore uniformly in λ $(\alpha < \lambda \leq \frac{1}{2}(\alpha + \beta))$

$$(7.1.9) \qquad \gamma = \sum_{h=n}^{m+n-1} s_h(\lambda) + O\,|\omega|^{-q}.$$

In the same way we find the same result in the interval

$$\tfrac{1}{2}(\alpha + \beta) \leq \lambda < \beta,$$

so that (7.1.9) holds uniformly in λ in the whole interval $\alpha < \lambda < \beta$. By definition the sum occurring on the right-hand side of (7.1.9) represents the function $\nu(\lambda)$ occurring in (7.1.1) which is negligible in the asymptotic neutrix N. Consequently for each real q it is possible to find a function

$\nu(\xi)$ negligible in N such that the order relation

(7.1.10) $$\gamma = \nu(\xi) + O\,|\omega|^{-q}$$

holds uniformly in ξ $(\alpha<\xi<\beta)$. This implies $\gamma \sim 0$ according to the asymptotic neutrix condition imposed on N. This completes the proof.

Proof that P is compatible with each smooth neutrix N with the same domain $\alpha<\xi<\infty$. The proof is the same as the preceding one with the difference that $O\,|\omega|^{-q}$ is everywhere replaced by $o(1)$, where $o(1)$ denotes a function of ξ which tends to zero as $\xi \to \infty$; here

$$\alpha < \lambda \leq \xi \leq \lambda + \rho$$

so that λ tends with ξ to infinity. In this way we find instead of (7.1.10)

$$\gamma = \nu(\xi) + o(1)$$

where $\nu(\xi)$ is negligible in the smooth neutrix N. From the definition of a smooth neutrix given at the beginning of this section, it follows that the function $o(1)$ is also negligible in N. Consequently $\nu(\xi) + o(1)$ is negligible in N, hence $\gamma = 0$. This establishes the proof.

7.2. Hadamard neutrices.

Definition: Let a be a complex number. Let K be a continuous curve lying in the complex ξ-plane or on a Riemann surface. We distinguish four cases, where $\arg(\xi - a)$ is bounded for each point ξ on K.

(1) a is the initial point of K and does not belong to this curve. The Hadamard neutrix H_{a+} with domain K is formed by the functions $\nu(\xi)$ defined on K of the form

(7.2.1) $$\nu(\xi) = \lambda(\xi) + o(1)$$

where $o(1)$ denotes a function of ξ which tends to zero as ξ tends on K to the initial point a; furthermore $\lambda(\xi)$ is a linear combination with constant coefficients, of functions of the form

(7.2.2) $$(\xi - a)^{\sigma} \log^{k}(\xi - a)$$

where the exponents σ are arbitrary complex constants and where the exponents k are arbitrary constant integers $\geqq 0$ with the understanding that the couple $\sigma = 0$, $k = 0$ is not admitted.

(2) a does not lie on K and the end point of this curve lies at infinity; this means that $|\xi| \to \infty$ as ξ approaches on K the end point of the curve. The Hadamard neutrix $H_\infty(a)$ with domain K is formed by the functions $\gamma(\xi)$ defined on K of the form (7.2.1), where $o(1)$ denotes a function of ξ which tends to zero as ξ approaches on K the end point of the curve; $\lambda(\xi)$ is defined in the same way as in case (1).

(3) a is the end point of K and does not belong to K. The Hadamard neutrix H_{a-} with domain K is formed by the functions $\gamma(\xi)$ defined on K of the form (7.2.1), where $o(1)$ denotes a function of ξ which tends to zero as ξ tends on K to the end point a; $\lambda(\xi)$ is a linear combination, with constant coefficients, of functions of the form

$$(a-\xi)^\sigma \log^k (a-\xi)$$

where the exponents σ are arbitrary complex constants and where the exponents are arbitrary constant integers $\geqq 0$, with the understanding that the couple $\sigma = 0$, $k = 0$ is not admitted.

(4) a does not lie on K and the initial point of the curve K lies at infinity. The Hadamard neutrix $H_{-\infty}(a)$ with domain K is formed by the functions $\gamma(\xi)$ defined on K of the form (7.2.1), where $o(1)$ denotes a function of ξ which tends to zero as ξ approaches on K the initial point of this curve; $\lambda(\xi)$ is defined in the same way as in case (3).

To show that each of these four sets H_{a+}, $H_\infty(a)$, H_{a-} and $H_{-\infty}(a)$ satisfies the neutrix condition, I assume that for each point ξ on K

$$\sum_{\nu=1}^{n} \sum_{\mu=0}^{m_\nu-1} c_{\nu\mu}(\xi-a)^{\sigma_\nu} \log^\mu (\xi-a) + o(1) = \gamma, \tag{7.2.3}$$

where n, $m_1, \ldots, m_n$, $c_{\nu\mu}$, σ_ν and γ are independent of ξ. We shall show that H_{a+} satisfies the neutrix condition; the proof for $H_\infty(a)$, H_{a-} and $H_{-\infty}(a)$ runs in the same way. Here $o(1)$ denotes a function of ξ which tends to zero as ξ tends on K to a. We must prove that $\gamma = 0$. Without loss of generality we may assume that the n exponents $\sigma_1, \ldots, \sigma_n$ are distinct and that each $\operatorname{Re}\sigma_\nu$ is $\leqq 0$, since the possible terms with $\operatorname{Re}\sigma_\nu > 0$ can be incorporated into the term $o(1)$. Moreover we may assume that $\sigma_1 = 0$, since we may add terms with coefficient $= 0$. The term with $\nu = 1$, $\mu = 0$ does not occur in (7.2.3) since the couple $\sigma = 0$, $k = 0$ is not

admitted. We can therefore write (7.2.3) in the form

$$\sum_{\nu=1}^{n} \sum_{\mu=0}^{m_\nu-1} c_{\nu\mu} (\xi - a)^{\sigma_\nu} \log^\mu (\xi - a) = o(1),$$

where $c_{10} = -\gamma$.

For each point w tending on K to a it is possible to find a positive number $\varepsilon = \varepsilon(w)$ tending to zero such that for each point ξ lying on K between a and w

$$\sum_{\nu=1}^{n} \sum_{\mu=0}^{m_\nu-1} c_{\nu\mu} (\xi - a)^{\sigma_\nu} \log^\mu (\xi - a) = O\varepsilon. \tag{7.2.4}$$

Let λ be a fixed positive number $< \dfrac{1}{r-1}$, where $r = m_1 + \ldots + m_n$. If ξ traverses an arc of K with endpoint w and diameter $|w-a|\varepsilon^\lambda$, then $\zeta = \dfrac{\xi - w}{w-a}$ traverses a continuous curve K^* with endpoint 0 and diameter ε^λ. Then $\xi - a = (w-a)(1+\zeta)$, so that (7.2.4) assumes the form

$$\sum_{\nu=1}^{n} \sum_{\mu=0}^{m_\nu-1} \gamma_{\nu\mu} (1+\zeta)^{\sigma_\nu} \log^\mu (1+\zeta) = O\varepsilon \tag{7.2.5}$$

where

$$\gamma_{\nu\mu} = (w-a)^{\sigma_\nu} \sum_{h=\mu}^{m_\nu-1} \binom{h}{\mu} c_{\nu h} \log^{h-\mu}(w-a)$$

in particular, in virtue of $\sigma_1 = 0$,

$$\gamma_{1\mu} = \sum_{h=\mu}^{m_1-1} \binom{h}{\mu} c_{1h} \log^{h-\mu}(w-a). \tag{7.2.6}$$

We apply the lemmas of the next section. According to Lemma 3 it is possible to find on K^* r points $\alpha_0, \alpha_1, \ldots, \alpha_{r-1}$ such that

$$\prod_{0 \leq v < u < r} |\alpha_u - \alpha_v|$$

has at least the same order of magnitude as $\varepsilon^{\frac{1}{2}r(r-1)\lambda}$. According to the remark added to Lemma 2 the determinant $\Delta((1+\alpha_\rho)^{\sigma_\nu} \log^\mu(1+\alpha_\rho))$ has

then at least the same order of magnitude as $\varepsilon^{\frac{1}{2}r(r-1)\lambda}$, whereas each minor in this determinant is $O\varepsilon^{\frac{1}{2}(r-1)(r-2)\lambda}$. Consequently it follows from (7.2.5), applied with $\zeta = \alpha_\rho$ $(\rho = 0, 1, ..., r-1)$, that

$$\gamma_{\nu\mu} = O\varepsilon^{1+\frac{1}{2}(r-1)(r-2)\lambda - \frac{1}{2}r(r-1)\lambda} = O\varepsilon^{1-(r-1)\lambda}$$

where the last exponent is positive, so that $\gamma_{\nu\mu} \to 0$ as w tends on K to a. Using (7.2.6) we see that for $0 \leq \mu < m_1$

$$\sum_{h=\mu}^{m_1-1} \binom{h}{\mu} c_{1h} \log^{h-\mu}(w-a) \to 0 \; .$$

Here $\log(w-a)$ tends in absolute value to infinity, so that each constant c_{1h} is equal to zero, in particular $\gamma = -c_{10} = 0$. This completes the proof.

7.3. Asymptotic Hadamard neutrices.

Definition : Let K be a continuous curve depending on ω which lies in the complex plane or on a Riemann surface. Let a be a point which does not lie on K and let w be a point lying on K such that for each fixed positive ε

(7.3.1) $$|w-a| = O\,|\omega|^{\varepsilon}\, d \quad \text{and} \quad \log(w-a) = O\,|\omega|^{\varepsilon},$$

where d denotes the diameter of K.

The asymptotic neutrix A_{a+} with domain K is formed by the functions $\nu(\xi)$ defined on K which for each real q can be written as

$$\lambda(\xi) + O\,|\omega|^{-q}$$

where $\lambda(\xi)$ is a linear combination of functions of the form (7.2.2); the number of terms occurring in $\lambda(\xi)$ and the exponents σ and k are independent of ω and ξ, but may depend on q; the exponents k are integers $\geqq 0$ and the couple $\sigma = 0$, $k = 0$ is not admitted. The coefficients occurring in the linear combination $\lambda(\xi)$ may depend on ω and q, but not on ξ.

A_{a-} is defined in a similar way, but in the definition of this asymptotic neutrix we use instead of (7.2.2) the functions

$$(a-\xi)^{\sigma} \log^{k}(a-\xi) \, .$$

In this section we shall show that A_{a+} and A_{a-} satisfy the asymptotic neutrix condition. The proof is similar to that given for the Hadamard neutrices H_{a+} and H_{a-}.

Lemma 1: Let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be arbitrary numbers and let $m_1, m_2, \ldots, m_n$ be arbitrary positive integers. Assume $z \neq 0$. Consider the determinant

$$\Delta\,(z^{\sigma_\nu} \log^\mu z)^{(\rho)}$$

formed by $r^2 = (m_1 + m_2 + \ldots + m_n)^2$ elements $(1 \leqq \nu \leqq n;\ 0 \leqq \mu < m_\nu;\ 0 \leqq \rho < r)$, where $(z^{\sigma_\nu} \log^\mu z)^{(\rho)}$ denotes the ρ^{th} derivative with respect to z of $z^{\sigma_\nu} \log^\mu z$. Then

$$\Delta\,(z^{\sigma_\nu} \log^\mu z)^{(\rho)} = \Gamma\, z^{m_1 \sigma_1 + \ldots + m_n \sigma_n - \frac{1}{2} r(r-1)} \tag{7.3.2}$$

where

$$\Gamma = \left(\prod_{\nu=1}^{n} 1\,!\; 2\,! \ldots (m_\nu - 1)\,!\right) \prod_{1 \leqq \lambda < \nu \leqq n} (\sigma_\nu - \sigma_\lambda)^{m_\nu m_\lambda}.$$

Proof: Let us first treat the special case that $m_1 = m_2 = \ldots = m_n = 1$. Then $r = n$ and the determinant Δ is equal to

$$z^{\sigma_1 + \ldots + \sigma_n - \frac{1}{2} n(n-1)}\, \Delta\,(\sigma_\nu(\sigma_\nu - 1) \ldots (\sigma_\nu + 1 - \rho))$$

where the last determinant is equal to the determinant of Vandermonde

$$\Delta\,(\sigma_\nu^\rho) = \prod_{1 \leqq \lambda < \nu \leqq n} (\sigma_\nu - \sigma_\lambda).$$

This gives the required result for the special case $m_1 = m_2 = \ldots = m_n = 1$.

In the general case I put

$$\tau_{\nu\mu} = \sigma_\nu + \mu\varepsilon_\nu \qquad (1 \leqq \nu \leqq n;\ 0 \leqq \mu < m_\nu)$$

where $\varepsilon_\nu \neq 0$, and we apply the result obtained above with the n numbers $\sigma_1, \ldots, \sigma_n$ replaced by the $m_1 + m_2 + \ldots + m_n$ numbers $\tau_{\nu\mu}$. This yields

$$\Delta\,(z^{\tau_{\nu\mu}})^{(\rho)} = z^{\Sigma \tau_{\nu\mu} - \frac{1}{2} r(r-1)} \Bigg(\prod_{\substack{1 \leqq \lambda < \nu \leqq n \\ 0 \leqq \mu < m_\nu \\ 0 \leqq \kappa < m_\lambda}} (\tau_{\nu\mu} - \tau_{\lambda\kappa})\Bigg) \prod_{\substack{\nu=1 \\ 0 \leqq \kappa < \mu < m_\nu}}^{n} (\tau_{\nu\mu} - \tau_{\nu\kappa})$$

where the sum $\Sigma \tau_{\nu\mu}$ is extended over the integers ν and μ with $1 \leqq \nu \leqq n;\ 0 \leqq \mu < m_\nu$. The last product is equal to

$$\prod_{\substack{\nu=1 \\ 0 \leqq \kappa < \mu < m_\nu}}^{n} (\mu - \kappa)\,\varepsilon_\nu = \left(\prod_{\nu=1}^{n} \varepsilon_\nu^{\frac{1}{2} m_\nu (m_\nu - 1)}\right) \left(\prod_{\nu=1}^{n} 1\,!\; 2\,! \ldots (m_\nu - 1)\,!\right).$$

If we put for $1 \leqq \nu \leqq n$, $0 \leqq \mu < m_\nu$

$$\chi_{\nu 0}(z) = z^{\tau_{\nu 0}}$$
$$\chi_{\nu 1}(z) = \varepsilon_\nu^{-1}(z^{\tau_{\nu 1}} - z^{\tau_{\nu 0}})$$
$$\chi_{\nu 2}(z) = \varepsilon_\nu^{-2}(z^{\tau_{\nu 2}} - 2z^{\tau_{\nu 1}} + z^{\tau_{\nu 0}})$$
$$\chi_{\nu 3}(z) = \varepsilon_\nu^{-3}(z^{\tau_{\nu 3}} - 3z^{\tau_{\nu 2}} + 3z^{\tau_{\nu 1}} - z^{\tau_{\nu 0}})$$

and so on, then the preceding results yield

$$\Delta\chi_{\nu\mu}^{(\rho)}(z) = \left(\prod_{\nu=1}^{n} \varepsilon_\nu^{-\frac{1}{2}m_\nu(m_\nu-1)}\right) \Delta\,(z^{\tau_{\nu\mu}})^{(\rho)}$$

$$= z^{\Sigma\tau_{\nu\mu} - \frac{1}{2}r(r-1)} \left(\prod_{\substack{1 \leqq \lambda < \nu \leqq n \\ 0 \leqq \mu < m_\nu \\ 0 \leqq \kappa < m_\lambda}} (\tau_{\nu\mu} - \tau_{\lambda\kappa})\right) \left(\prod_{\nu=1}^{n} (1\,!\,2\,!\,\ldots\,(m_\nu - 1)\,!)\right).$$

If the numbers $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ tend to zero, then the right-hand side tends to the right-hand side of (7.3.2) and

$$\lim \chi_{\nu\mu}^{(\rho)}(z) = \lim \left(\frac{\partial}{\partial z}\right)^{\rho} \chi_{\nu\mu}(z) = \left(\frac{\partial}{\partial z}\right)^{\rho} \lim \chi_{\nu\mu}(z) = \left(\frac{\partial}{\partial z}\right)^{\rho} z^{\sigma_\nu} \log^\mu z$$

according to the definition of the functions χ. This yields the required result.

The following lemma involves a positive integer r; in this lemma the notation $f = Og$ means that it is possible to find a positive number γ depending only on r such that $|f| \leqq \gamma\,|g|$. Furthermore $\Delta\chi_h(\alpha_\rho)$ represents the determinant formed by the r^2 components $\chi_h(\alpha_\rho)$ where $0 \leqq h < r$ and $0 \leqq \rho < r$.

Lemma 2: If the functions $\chi_h(z)$ $(0 \leqq h < r)$ are analytic functions of z for $|z| \leqq p \leqq 1$ with

$$|\chi_h^{(\rho)}(z)| \leqq c \qquad (0 \leqq h < r\,;\ 0 \leqq \rho < r)$$

then we have for each choice of the points $\alpha_0, \alpha_1, \ldots, \alpha_{r-1}$ with $|\alpha_\rho| \leqq p$

$$\Delta\chi_h(\alpha_\rho) = \frac{\Gamma}{1\,!\,2\,!\,\ldots\,(r-1)\,!} \prod_{0 \leqq v < u < r} (\alpha_u - \alpha_v) + Oc^r\,\beta^{\frac{1}{2}r(r-1)+1}$$

where

$$\beta = \max_{\rho} |\alpha_\rho| \quad \text{and} \quad \Gamma = \Delta \chi_h^{(\rho)}(0).$$

Proof: Considerations of homogeneity show that without loss of generality we can assume that $c = 1$, for otherwise we can replace $\chi(z)$ by $c^{-1}\chi(z)$. We have

$$\chi_h(\alpha_\rho) = \sum_{k=0}^{r-1} \frac{1}{k!} \chi_h^{(k)}(0)\, \alpha_\rho^k + O\,\alpha_\rho^r \qquad (0 \leq h < r).$$

We can therefore write $\Delta\chi_h(\alpha_\rho)$ as a sum of determinants such that the number of these determinants depends only on r and that one of these determinants is equal to

$$\Delta\left(\sum_{k=0}^{r-1} \frac{1}{k!} \chi_h^{(k)}(0)\, \alpha_\rho^k\right) = \left(\Delta \frac{1}{k!} \chi_h^{(k)}(0)\right) \Delta\, \alpha_\rho^k$$

$$= \frac{\Gamma}{1!\,2!\,\dots(r-1)!} \prod_{0 \leq v < u < r} (\alpha_u - \alpha_v).$$

Each other determinant Δ^* occurring in this sum contains columns of two different kinds. If the $(\rho + 1)^{\text{st}}$ column of Δ^* is a column of the first kind, then it is formed by the numbers

$$\frac{1}{k!} \chi_h^{(k)}(0)\, \alpha_\rho^k \qquad (0 \leq k < r).$$

If the $(\rho+1)^{\text{st}}$ column of Δ^* is a column of the second kind, then each of its elements is $O\,\alpha_\rho^r = O\,\beta^r$. Each determinant Δ^* contains at least one column of the second kind. The number q of the columns of the first kind occurring in Δ^* is therefore $\geqq 0$ and $\leqq r-1$. Let us denote the numbers k corresponding to these columns by $k_1, k_2, \dots, k_q$. If two or more of these q integers are equal, then Δ^* has two equal columns and is therefore equal to zero. If the integers $k_1, k_2, \dots, k_q$, each $\geqq 0$, are distinct, then

$$k_1 + k_2 + \dots + k_q \geqq 1 + 2 + \dots + (q-1) = \tfrac{1}{2}\, q\,(q-1)$$

so that

$$\Delta^* = O\,\beta^{\frac{1}{2}q(q-1) + r(r-q)}.$$

If q is augmented by 1, then

$$\tfrac{1}{2}\, q\,(q-1) + r\,(r-q)$$

is augmented by $q - r < 0$, so that it assumes its smallest value at $q = r - 1$. In this way we find that each of the determinants Δ^* is

$$O\, \beta^{\frac{1}{2}(r-1)(r-2)+r} = O\, \beta^{\frac{1}{2} r(r-1)+1}$$

which gives the required result.

Remark: Choosing in this lemma $p = \frac{1}{2}$ and replacing $\chi_h(\alpha_\rho)$ by

(7.3.3) $\quad \chi_{\nu\mu}(\alpha_\rho) = (1+\alpha_\rho)^{\sigma_\nu} \log^\mu (1+\alpha_\rho) \quad (1 \leqq \nu \leqq n \;;\; 0 \leqq \mu < m_\nu)$

then we find for suitably chosen c depending only on n, σ_ν and m_ν $(1 \leqq \nu \leqq n)$

(7.3.4) $$\Delta(1+\alpha_\rho)^{\sigma_\nu} \log^\mu (1+\alpha_\rho)$$

$$= \frac{\Gamma}{1!\; 2! \ldots (r-1)!} \prod_{0 \leqq v < u < r} (\alpha_u - \alpha_v) + O c^r\, \beta^{\frac{1}{2} r(r-1)+1}.$$

Let ν_1 be a positive integer $\leqq n$; let μ_1 be an integer $\geqq 0$ and $< m_{\nu_1}$. If we replace in Lemma 2, r by $r-1$, if we choose $p = \frac{1}{2}$ and if we define $\chi_{\nu\mu}(\alpha_\rho)$ by (7.3.3), where we leave out of consideration the function corresponding to $\nu = \nu_1$ and $\mu = \mu_1$, then we find that in the determinant (7.3.4) the minor of

$$(1+\alpha_\rho)^{\sigma_{\nu_1}} \log^{\mu_1} (1+\alpha_\rho) \quad \text{is} \quad O c_1^{r-1}\, \beta^{\frac{1}{2}(r-1)(r-2)}$$

where c_1 denotes a suitably chosen positive number depending only on n, σ_ν and m_ν $(1 \leqq \nu \leqq n)$.

Lemma 3: *On each continuous curve K with diameter d it is possible to find r points $\alpha_0, \alpha_1, \ldots, \alpha_{r-1}$ such that*

(7.3.5) $$\prod_{0 \leqq v < u < r} |\alpha_u - \alpha_v| \geqq \left(\frac{d}{r}\right)^{\frac{1}{2} r(r-1)}.$$

Proof: We can find on K two points α_0 and α_{r-1} with

$$|\alpha_{r-1} - \alpha_0| = \frac{r-1}{r}\, d.$$

Then we can find on K the points $\alpha_1, \ldots, \alpha_{r-2}$ such that

$$|\alpha_\rho - \alpha_0| = \frac{\rho d}{r} \quad (\rho = 1, 2, \ldots, r-2)$$

hence

$$|\alpha_u - \alpha_v| \geq |\alpha_u - \alpha_0| - |\alpha_v - \alpha_0| \geq \frac{ud}{r} - \frac{vd}{r} \geq \frac{d}{r}$$

which yields the required result.

Proof that A_{a+} satisfies the asymptotic neutrix condition (for A_{a-} the proof runs in the same way). Assume that a number γ independent of ξ can be written for each fixed real q in the form

$$\gamma = \sum_{\nu=1}^{n} \sum_{\mu=0}^{m_\nu - 1} c_{\nu\mu} (\xi - a)^{\sigma_\nu} \log^\mu (\xi - a) + O|\omega|^{-q}$$

uniformly in ξ $(\alpha < \xi < \beta)$; here n, m_ν and σ_ν may depend on q and the coefficients $c_{\nu\mu}$ may depend on ω and q. We must show that γ is asymptotically equal to zero. Without loss of generality we can assume that the numbers $\sigma_1, \ldots, \sigma_n$ are distinct. We may also assume that $\sigma_1 = 0$, since we may always add terms with coefficient zero. In view of the fact that the term with $\sigma_1 = 0$, $\mu = 0$ does not occur on the right-hand side, I may therefore write

$$\sum_{\nu=1}^{n} \sum_{\mu=0}^{m_\nu - 1} c_{\nu\mu} (\xi - a)^{\sigma_\nu} \log^\mu (\xi - a) = O|\omega|^{-q} \tag{7.3.6}$$

where $c_{10} = -\gamma$. Put $m_1 + m_2 + \ldots + m_n = r$.

I treat first the special case that the point w occurring in the definition of A_{a+} coincides with $a + 1$. By hypothesis $d^{-1} = d^{-1}|w - a|$ is $O|\omega|^\varepsilon$ for each fixed positive ε. We can therefore find a fixed positive number $\delta < r^{-1}$ such that K has an arc with diameter $|\omega|^{-\delta}$ which contains the point $w = a + 1$. According to Lemma 3 we find r points α_ρ $(0 \leq \rho < r)$ such that

$$\prod_{0 \leq v < u < r} |\alpha_u - \alpha_v| \geq \left|\frac{\omega^{-\delta}}{r}\right|^{\frac{1}{2} r(r-1)}$$

and that the r points

$$\xi_\rho = a + 1 + \alpha_\rho \qquad (0 \leq \rho < r)$$

lie on this arc. Then the first term on the right-hand side of (7.3.4) has at least the same order of magnitude as $|\omega|^{-\frac{1}{2}\delta r(r-1)}$ and the second term

on the right-hand side of (7.3.4) is $O\,|\,\omega\,|^{-\frac{1}{2}\delta r(r-1)+\delta}$ so that

$$\Delta\,(1+\alpha_\rho)^{\sigma_\nu}\log^\mu(1+\alpha_\rho)$$

has at least the same order of magnitude as $|\,\omega\,|^{-\frac{1}{2}\delta r(r-1)}$. In the remark added to Lemma 2 we have seen that in this determinant the minor of $(1+\alpha_\rho)^{\sigma_\nu}\log^\mu(1+\alpha_\rho)$ is $O\,|\,\omega\,|^{-\frac{1}{2}\delta(r-1)(r-2)}$. Consequently it follows from (7.3.6) that

$$c_{\nu\mu}=O\,|\,\omega\,|^{-q+\delta r}=O\,|\,\omega\,|^{1-q}\quad(1\leqq\nu\leqq n\,;\;0\leqq\mu<m_\nu). \tag{7.3.7}$$

In particular

$$\gamma=O\,|\,\omega\,|^{1-q}$$

and this holds for each fixed real q, so that γ is asymptotically equal to zero.

Let us now treat the general case. The substitution $\xi-a=(w-a)\zeta$ transforms (7.3.6) into

$$\sum_{\nu=1}^{n}\sum_{\mu=0}^{m_\nu-1}\gamma_{\nu\mu}\,\zeta^{\sigma_\nu}\log^\mu\zeta=O\,|\,\omega\,|^{-q}$$

where

$$\gamma_{\nu\mu}=(w-a)^{\sigma_\nu}\sum_{h=\mu}^{m_\nu-1}\binom{h}{\mu}c_{\nu\mu}\log^{h-\mu}(w-a). \tag{7.3.8}$$

If ξ traverses K, then ζ traverses a curve K^* which contains the point 1. The substitution transforms a into 0. We can apply (7.3.7) with K, a, w, $c_{\nu\mu}$ replaced by K^*, 0, 1, $\gamma_{\nu\mu}$, since the diameter d^* of K^* is equal to $|\,w-a\,|^{-1}d$, so that according to (7.3.1) for each fixed positive number ε the relation

$$|\,1-0\,|=1=O\,|\,\omega\,|^{\varepsilon}\,|\,w-a\,|^{-1}d=O\,|\,\omega\,|^{\varepsilon}d^*$$

holds. In this way we obtain for $0\leqq\mu<m_1$

$$\sum_{h=\mu}^{m_1-1}\binom{h}{\mu}c_{1h}\log^{h-\mu}(w-a)=\gamma_{1\mu}=O\,|\,\omega\,|^{1-q}. \tag{7.3.9}$$

$$\log(w-a) = O\,|\,\omega\,|^{\varepsilon}, \quad \text{where} \quad \varepsilon < \frac{1}{m_1 - 1}.$$

Formula (7.3.9) yields for $0 \leqq \mu < m_1$

$$c_{1\mu} = O\,|\,\omega\,|^{1-q+(m_1-1-\mu)\varepsilon}. \tag{7.3.10}$$

This follows for $\mu = m_1 - 1$ from (7.3.9); if $0 \leqq \mu < m_1 - 1$ and (7.3.10) holds with μ replaced by $\mu+1$, $\mu+2$, ..., $m_1 - 1$, then (7.3.9) yields (7.3.10). Consequently

$$\gamma = -c_{10} = O\,|\,\omega\,|^{2-q}$$

for each fixed real q, so that γ is asymptotically equal to zero.

7.4. $H_\infty(a)$ **and** $H_\infty(b)$ **have different properties.** In Theorem 3.1.1 we have found for each real a and for each integer $k \geqq 0$ that

$$\int_{H_{a+}}^{H_\infty(a)} (x-a)^{\sigma-1} \log^k (x-a)\, dx = 0. \tag{7.4.1}$$

In this section I prove a more general result, namely

Theorem 7.4.1: *If a does not lie on the path of the following integral, then*

$$\int_{H_{a+}}^{H_\infty(b)} (x-a)^{\sigma-1} \log^k (x-a)\, dx \tag{7.4.2}$$

is equal to zero if σ is not a positive integer and it is in the case that σ is a positive integer equal to

$$\frac{(b-a)^\sigma}{\sigma\,!} \left\{ \left(\frac{d}{ds} \right)^k (s-1)(s-2)\ldots(s+1-\sigma) \right\}_{s=\sigma}. \tag{7.4.3}$$

Remark: In the special case $b=0$ the integral (7.4.2) is called by Hadamard the finite part of the integral

$$\int_a^\infty (x-a)^{\sigma-1} \log^k (x-a)\, dx.$$

The choice $b=0$ is rather artificial and often another choice gives a simpler result. For instance in the case $b = a$ the integral (7.4.2) is according to (7.4.1) equal to zero for each choice of σ.

Proof: We have

$$(7.4.4) \qquad \int_{H_{a+}}^{\xi} (x-a)^{\sigma-1} \log^k (x-a)\, dx = \psi(\xi-a, \sigma, k)$$

where ψ denotes the function defined in (3.1.5). Let ξ denote the variable of the neutrix $H_\infty(b)$. We have

$$\log(\xi-a) = \log(\xi-b) + O\left(\frac{1}{\xi-b}\right)$$

so that

$$\log^{k+1}(\xi-a) = \log^{k+1}(\xi-b) + o(1)$$

where $o(1)$ tends to zero as $|\xi| \to \infty$. Consequently $\log^{k+1}(\xi-a)$ is negligible in $H_\infty(b)$, so that (7.4.2) vanishes for $\sigma = 0$. If σ is not an integer $\geqq 0$, then for each integer $j \geqq 0$

$$(\xi-a)^\sigma \log^j (\xi-a)$$

can be written in the form $\lambda(\xi) + o(1)$, where $o(1)$ tends to zero as $|\xi| \to \infty$ and where $\lambda(\xi)$ is a linear combination, with constant coefficients, of functions of the form

$$(\xi-b)^{\sigma-h} \log^{\mu} (\xi-b)$$

where the numbers h are integers $\geqq 0$ and the numbers μ are integers $\geqq 0$ and $\leqq j$. Since σ is not an integer $\geqq 0$, the exponents $\sigma - h$ are $\neq 0$, so that (7.4.4) vanishes at $\xi = H_\infty(b)$.

Let us finally consider the case that σ is a positive integer. For sufficiently large positive integer l we have

$$\int_{H_{a+}}^{\xi} (x-a)^{\sigma-1}\, dx = \sigma^{-1} (\xi-a)^\sigma = \sigma^{-1} \sum_{h=0}^{l-1} \binom{\sigma}{h} (b-a)^h (\xi-b)^{\sigma-h} + o(1).$$

All the terms occurring on the right-hand side are negligible in $H_\infty(b)$, except the constant term which is equal to

$$\sigma^{-1} (b-a)^\sigma = \frac{(b-a)^\sigma}{\sigma!} (\sigma-1)(\sigma-2) \ldots 1.$$

This gives the required result (7.4.3) in the special case $k = 0$. Assume

that k is $\geqq 1$ and that the assertion has already been proved with k replaced by $k-1$.

For $|\xi - b| > |b-a|$ we have

$$(\xi - a)^\sigma = \sum_{h=0}^{\infty} \binom{\sigma}{h} (b-a)^h (\xi - b)^{\sigma - h}.$$

Differentiation k times with respect to σ gives

$$(\xi - a)^\sigma \log^k (\xi - a) = \sum_{h=0}^{\infty} (b-a)^h (\xi - b)^{-h} \left\{ \left(\frac{d}{d\sigma} \right)^k \binom{\sigma}{h} (\xi - b)^\sigma \right\}.$$

We obtain the constant term occurring in this expansion by choosing $h = \sigma$ and by using in

$$\left\{ \left(\frac{d}{d\sigma} \right)^k \binom{\sigma}{h} (\xi - b)^\sigma \right\} = \sum_{n=0}^{k} \binom{k}{n} \left\{ \left(\frac{d}{d\sigma} \right)^n \binom{\sigma}{h} \right\} \frac{d^{k-n} (\xi - b)^\sigma}{d\sigma^{k-n}}$$

only the term with $n = k$. The said constant term is therefore equal to

$$(b-a)^\sigma \left\{ \left(\frac{d}{d\sigma} \right)^k \binom{\sigma}{h} \right\}_{h=\sigma} = (b-a)^\sigma \left\{ \left(\frac{d}{ds} \right)^k \binom{s}{\sigma} \right\}_{s=\sigma}.$$

We write (7.4.4) as

$$\sigma^{-1} (\xi - a)^\sigma \log^k (\xi - a) - k\sigma^{-1} \int_{H_{a+}}^{\xi} (x-a)^{\sigma-1} \log^{k-1} (x-a)\, dx$$

hence

$$(7.4.5) \quad \int_{H_{a+}}^{H\infty(b)} (x-a)^{\sigma-1} \log^k (x-a)\, dx = \sigma^{-1} (b-a)^\sigma \left\{ \left(\frac{d}{ds} \right)^k \binom{s}{\sigma} \right\}_{s=\sigma}$$

$$- k\sigma^{-1} \int_{H_{a+}}^{H\infty(b)} (x-a)^{\sigma-1} \log^{k-1} (x-a)\, dx$$

$$= \frac{(b-a)^\sigma}{\sigma!\ \sigma} \left\{ \left(\frac{d}{ds} \right)^k s(s-1) \ldots (s+1-\sigma) \right\}_{s=\sigma}$$

$$- k \frac{(b-a)^\sigma}{\sigma!\ \sigma} \left\{ \left(\frac{d}{ds} \right)^{k-1} (s-1)(s-2) \ldots (s+1-\sigma) \right\}_{s=\sigma}$$

according to the induction hypothesis. This gives the required result (7.4.3) since the first expression occurring on the right-hand side of (7.4.5) between the braces is according to the formula of Leibniz equal to

$$\sigma \left\{ \left(\frac{d}{ds} \right)^k (s-1)(s-2)\dots(s+1-\sigma) \right\}_{s=\sigma}$$
$$+ k \left\{ \left(\frac{d}{ds} \right)^{k-1} (s-1)(s-2)\dots(s+1-\sigma) \right\}_{s=\sigma} .$$

7.5. The value of a Hadamard integral at a singular point. Under the conditions of Theorem 3.4.1 the function

$$\chi(s) = \int_{H_{a+}}^{b} g^{-s}(x)\, f(x)\, dx \tag{7.5.1}$$

is analytic in the whole complex s-plane, the points $\sigma_h + \tau_k$ $(h \geqq 0,\ k \geqq 0)$ excepted. These points $\sigma_h + \tau_k$ are poles of $\chi(s)$. In the particular case that $g(x) = x - a$ the function $\chi(s)$ is analytic in the whole complex s-plane, the points $\sigma_0, \sigma_1, \dots$ excepted; each point σ occurring in the sequence $\sigma_0, \sigma_1, \dots$ is a pole of $\chi(s)$ and $\chi(\sigma)$ is the constant term in the Laurent expansion of $\chi(s)$ according to powers of $s - \sigma$.

The question arises whether it is possible to find a similar result in the general problem treated in the said theorem. That this is indeed possible follows from

Theorem 7.5.1: *If the conditions of Theorem 3.4.1 are satisfied and if σ occurs in the sequence $\sigma_h + \tau_k$ $(h \geqq 0,\ k \geqq 0)$ then σ is a pole of the function $\chi(s)$ defined in (7.5.1); furthermore*

$$\chi(\sigma) = \gamma + \sum_m \rho_m ; \tag{7.5.2}$$

γ is the constant term occurring in the Laurent expansion of $\chi(s)$ according to powers of $s - \sigma$; $\rho_m / m!$ is the coefficient of

$$(x-a)^{\sigma-1}(s-\sigma)^{m+1}\log^m(x-a) \tag{7.5.3}$$

in the asymptotic expansion of $(x-a)^s g^{-s}(x) f(x)$ according to powers of $x-a$, $\log(x-a)$ and $s-\sigma$; the sum $\sum_m$ is extended over the integers $m \geqq 0$ with $\rho_m \neq 0$; the number of integers m with this property is finite.

Proof: The function $g^{-s}(x) f(x)$ has for small positive $x-a$ and for the points s in the neighborhood of σ an asymptotic expansion of the form

$$g^{-s}(x) f(x) \sim \sum_{\overleftarrow{h,k}}{}' (x-a)^{\sigma_h+\tau_k-s-1} p_{hk}(\log(x-a), s-\sigma)$$

where $p_{hk}(u, v)$ is a polynomial in u and v. Consequently for the points s in the neighborhood of σ we have

$$\text{(7.5.4)} \qquad \int_{H_{a+}}^{b} g^{-s}(x) f(x)\, dx$$

$$= \int_{H_{a+}}^{b} (x-a)^{\sigma-s-1} \sum_{\sigma_h+\tau_k=\sigma} p_{hk}(\log(x-a), s-\sigma)\, dx + \mu(s)$$

where $\mu(s)$ is analytic at $s=\sigma$. Put

$$\text{(7.5.5)} \qquad \sum_{\sigma_h+\tau_k=\sigma} p_{hk}(u, v) = \sum \lambda_{mn} u^m v^n$$

where the sum is extended over a finite number of integers $m \geqq 0$ and $n \geqq 0$. According to (3.1.5)

$$\text{(7.5.6)} \qquad (s-\sigma)^n \int_{H_{a+}}^{b} (x-a)^{\sigma-s-1} \log^m(x-a)\, dx$$

$$= -m!\,(b-a)^{\sigma-s} \sum_{h=0}^{m} \frac{\log^{m-h}(b-a)}{(m-h)!\,(s-\sigma)^{h+1-n}}$$

so that this function has a Laurent expansion according to powers of $s-\sigma$ in which the constant term is equal to

$$(-)^n m! \log^{m+1-n}(b-a) \sum_h \frac{(-)^h}{(h+1-n)!\,(m-h)!};$$

the last sum is extended over the integers h with

$$\max(0, n-1) \leqq h \leqq m.$$

Putting $m - h = t$ we write the sum $\sum\limits_h$ in the form

$$\frac{(-)^m}{(m+1-n)!} \sum_t (-)^t \binom{m+1-n}{t}$$

where the sum $\sum\limits_t$ is extended over the integers t with

$$0 \leqq t \leqq \min(m, m+1-n).$$

Consequently, if $n \geqq 1$, then the said sum is equal to

$$\frac{(-)^m}{(m+1-n)!}(1-1)^{m+1-n} = \begin{cases} 0 & \text{for } n \leqq m \\ (-)^m & \text{for } n = m+1 \end{cases}$$

however, if $n = 0$, then the said sum is equal to

$$\frac{(-)^m}{(m+1)!}\{(1-1)^{m+1} - (-)^{m+1}\} = \frac{1}{(m+1)!}.$$

Using (7.5.5) and (7.5.6) we find therefore for the integral occurring on the right-hand side of (7.5.4) a Laurent expansion in which the constant term is equal to

$$-\sum_{m \geqq 0} m!\, \lambda_{m, m+1} + \sum_{m \geqq 0} \frac{\lambda_{m,0}}{m+1} \log^{m+1}(b-a);$$

both these sums are finite. The left-hand side of (7.5.4) represents therefore a function of s which possesses a Laurent expansion according to powers of $s - \sigma$ in which the constant term γ is equal to

$$-\sum_{m \geqq 0} m!\, \lambda_{m, m+1} + \sum_{m \geqq 0} \frac{\lambda_{m,0}}{m+1} \log^{m+1}(b-a) + \mu(\sigma).$$

Formula (7.5.4) holds also at $s = \sigma$ so that

$$\begin{aligned} \chi(\sigma) &= \sum_{m \geqq 0} \lambda_{m0} \int_{H_{a+}}^{b} (x-a)^{-1} \log^m (x-a)\, dx + \mu(\sigma) \\ &= \sum_{m \geqq 0} \frac{\lambda_{m,0}}{m+1} \log^{m+1}(b-a) + \mu(\sigma) \\ &= \gamma + \sum_{m \geqq 0} m!\, \lambda_{m, m+1}. \end{aligned}$$

Here $\lambda_{m,m+1}$ is by definition the coefficient of

$$(x-a)^{\sigma-1}(\log(x-a))^m(s-\sigma)^{m+1}$$

in the asymptotic expansion of $(x-a)^s g^{-s}(x) f(x)$ according to powers of $x-a$, $\log(x-a)$ and $s-\sigma$. Consequently $m!\,\lambda_{m,m+1}=\rho_m$, which gives the required result (7.5.2).

7.6. Introduction of a new integration variable.

Several times in the preceding chapters we have introduced new integration variables in the investigation of Hadamard integrals, but up till now only in very simple cases. In many applications we need more general theorems and it is the purpose of this section to deduce two such theorems.

Lemma 1: Let K be a continuous rectifiable curve with finite initial point α not belonging to the curve such that $\arg(\xi-\alpha)$ is bounded for each point ξ on K. Let $u(\xi)$ be a function $\neq 0$ defined on K such that for suitably chosen constant λ the function $(\xi-a)^{-\lambda}u(\xi)$ possesses at $\alpha+$ a normal Hadamard expansion with exponents $\tau_0,\tau_1,\ldots$. Then $u^\sigma(\xi)\log^k(\xi-a)$, where σ is an arbitrary complex constant and where k is an arbitrary constant integer $\geqq 0$, assumes a neutralized value at the Hadamard neutrix $H_{\alpha+}$ with domain K. If $-\lambda\sigma$ does not occur in the sequence $\tau_0,\tau_1,\ldots$, then this neutralized value is equal to zero.

Proof: According to the power theorem (Theorem 3.3.2), the function $u^\sigma(\xi)\log^k(\xi-a)$ possesses at $\alpha+$ a Hadamard expansion with exponents $\lambda\sigma+\tau_h$ $(h=0,1,\ldots)$. We can therefore write

$$u^\sigma(\xi)\log^k(\xi-a)=\Sigma_1 c_{hj}(\xi-\alpha)^{\lambda\sigma+\tau_h}\log^j(\xi-\alpha)+\rho(\xi) \tag{7.6.1}$$

where $\rho(\xi)\to 0$ as $\xi\to\alpha$ and where Σ_1 is a finite sum extended over integers $h\geqq 0$ and $j\geqq 0$. The function $\rho(\xi)$ is negligible in $H_{\alpha+}$. Each term occurring in the sum Σ_1 is negligible in $H_{\alpha+}$, unless $\lambda\sigma+\tau_h=0$; $j=0$. Consequently $u^\sigma(\xi)\log^k(\xi-a)$ assumes at $H_{\alpha+}$ the neutralized value $\Sigma_2 c_{h0}$, where Σ_2 is extended over the terms of Σ_1 with $\lambda\sigma+\tau_h=0$; $j=0$. If $-\lambda\sigma$ does not occur in the sequence $\tau_0,\tau_1,\ldots$, then the sum

Σ_2 is empty, so that then $u^\sigma(\xi)\log^k(\xi-a)$ assumes at $H_{\alpha+}$ the neutralized value 0.

Theorem 7.6.1: Assume that the conditions of the preceding lemma are satisfied, that K has a finite end point β belonging to the curve, that $\operatorname{Re}\lambda>0$ and that $u(\xi)$ is continuously differentiable along K.

If ξ traverses K, then $\eta = a+u(\xi)$ where a is given, traverses a curve K^* with initial point a not belonging to K^* and with end point $b=a+u(\beta)$ belonging to K^*.

Each function $f(y)$ integrable along K^* from $a+$ to b which possesses at $a+$ a Hadamard expansion with exponents $\sigma_0, \sigma_1, \ldots$ has the property that the integrals

$$(7.6.2) \qquad \int_{H_{\alpha+}}^{\beta} f(a+u(x))\,u'(x)\,dx \quad \text{and} \quad \int_{H_{a+}}^{b} f(y)\,dy$$

exist; the first integral is taken along K and $H_{\alpha+}$ is the Hadamard neutrix with domain K; the second integral is taken along K^* and H_{a+} is the Hadamard neutrix with domain K^*.

Finally, if the two sequences $-\lambda\sigma_h$ $(h=0, 1, \ldots)$ and τ_m $(m=0, 1, \ldots)$ have no element in common, then the two integrals occurring in (7.6.2) are equal.

Proof: For each point ξ on K we have

$$(7.6.3) \qquad \int_{\xi}^{\beta} f(a+u(x))\,u'(x)\,dx = \int_{a+u(\xi)}^{b} f(y)\,dy\,.$$

By hypothesis $f(y)$ satisfies on K^* a relation of the form

$$f(y) = \Sigma_1\, c_{hk}\,(y-a)^{\sigma_h-1}\log^k(y-a) + \rho(y)$$

where Σ_1 is a finite sum extended over integers $h\geqq 0$ and $k\geqq 0$ and where $\rho(y)\to 0$ as y tends on K^* to a. Consequently $\rho(y)$ is integrable along K^* from a to b. Using the function ψ defined in (3.1.5) we find for each point η on K^*

$$(7.6.4)\qquad \int_{\eta}^{b} f(y)\,dy = \Sigma_1\, c_{hk}\, \psi(b-a\,,\,\sigma_h\,,\,k) - \Sigma_1\, c_{hk}\, \psi(\eta-a\,,\,\sigma_h\,,\,k)$$

$$+ \int_{\eta}^{b} \rho(y)\,dy\,.$$

The second sum occurring on the right-hand side is negligible in H_{a+}; the last integral occurring in (7.6.4) assumes at H_{a+} the neutralized value

$$(7.6.5)\qquad \int_{a}^{b} \rho(y)\,dy$$

so that

$$(7.6.6)\qquad \int_{H_{a+}}^{b} f(y)\,dy = \Sigma_1\, c_{hk}\, \psi(b-a\,,\,\sigma_h\,,\,k) + \int_{a}^{b} \rho(y)\,dy\,.$$

Replacing in (7.6.4) η by $a+u(\xi)$ we find by means of (7.6.3)

$$(7.6.7)\qquad \int_{\xi}^{\beta} f(a+u(x))\,u'(x)\,dx$$

$$= \Sigma_1\, c_{hk}\, \psi(b-a\,,\,\sigma_h\,,\,k) - \Sigma_1\, c_{hk}\, \psi(u(\xi)\,,\,\sigma_h\,,\,k) + \int_{a+u(\xi)}^{b} \rho(y)\,dy\,.$$

The last integral assumes at $\xi = H_{\alpha+}$ the neutralized value (7.6.5), since $u(\xi) \to 0$ as ξ tends on K to α.

According to the preceding lemma $\psi(u(\xi)\,,\,\sigma_h\,,\,k)$ assumes at $H_{\alpha+}$ a neutralized value, so that also the left-hand side of (7.6.7) assumes a neutralized value at $H_{\alpha+}$. If the two sequences $-\lambda\sigma_h$ $(h = 0, 1, \ldots)$ and τ_m $(m = 0, 1, \ldots)$ have no element in common, then this neutralized value of ψ is equal to zero according to the preceding lemma, so that then

$$\int_{H_{\alpha+}}^{\beta} f(a+u(x))\,u'(x)\,dx = \Sigma_1\, c_{hk}\, \psi(b-a\,,\,\sigma_h\,,\,k) + \int_{a}^{b} \rho(y)\,dy\,.$$

This integral is therefore equal to the left-hand side of (7.6.6). This completes the proof.

In the same way we prove the following lemma and the following theorem.

Lemma 2: Let K be a continuous rectifiable curve with end point at infinity. Let α be a finite point not lying on K such that $\arg(\xi-\alpha)$ is bounded for each point ξ on the curve. Let $u(\xi)$ be a function $\neq 0$ defined on K such that for suitably chosen constant λ the function $(\xi-\alpha)^{-\lambda}u(\xi)$ possesses at infinity a Hadamard expansion with parameter α and with exponents $\tau_0, \tau_1, \dots$. Then $u^\sigma(\xi)\log^k(\xi-\alpha)$, where σ is an arbitrary complex constant and where k is an arbitrary constant integer $\geqq 0$, assumes a neutralized value at the Hadamard neutrix $H_\infty(\alpha)$ with domain K. If $-\lambda\sigma$ does not occur in the sequence $\tau_0, \tau_1, \dots$, then this neutralized value is equal to zero.

Theorem 7.6.2: Assume that the conditions of the preceding lemma are satisfied, that K has a finite initial point β belonging to K, that $\operatorname{Re}\lambda>0$ and that $u(\xi)$ is continuously differentiable along K.

If ξ traverses K, then $\eta=a+u(\xi)$ traverses a curve K^* with end point at infinity and with initial point $b=a+u(\beta)$ belonging to K^*.

Each function $f(y)$ integrable along K^* from b to ∞— which possesses at infinity a Hadamard expansion with parameter a and exponents $\sigma_0, \sigma_1, \dots$ has the property that the integrals

$$\int_\beta^{H_\infty(\alpha)} f(a+u(x))\,u'(x)\,dx \quad \text{and} \quad \int_b^{H_\infty(a)} f(y)\,dy$$

exist; if the two sequences $-\lambda\sigma_h\ (h=0, 1, \dots)$ and $\tau_m\ (m=0, 1, \dots)$ have no element in common, then these two integrals are equal.

Theorem 7.6.1 involves the neutrices $H_{\alpha+}$ and H_{a+}, whereas Theorem 7.6.2 involves the couple $H_\infty(\alpha)$ and $H_\infty(a)$. There are several theorems of the same kind, for instance a theorem involving the couple

$H_{\alpha-}$ and H_{a-}, or the couple $A_{\alpha+}$ and A_{a+}, or the couple H_{a+} and $H_\infty(a)$, and so on, but the formulation and the proofs of these theorems are so simple, that I leave this to the reader.

By choosing in Theorem 7.6.1

$$a = 0; \quad \lambda = 0; \quad u(\xi) = p\xi + q$$

where $p \neq 0$, we obtain as special case:

If $p \neq 0$ and q are real constants and if none of the exponents $\sigma_0, \sigma_1, \ldots$ is equal to zero, then

$$(7.6.8) \qquad p \int_{H_{\alpha+}}^{\beta} f(px+q)\,dx = \begin{cases} \displaystyle\int_{H_{p\alpha+q+}}^{p\beta+q} f(y)\,dy & \text{if } p>0 \\ \displaystyle -\int_{p\beta+q}^{H_{p\alpha+q-}} f(y)\,dy & \text{if } p<0. \end{cases}$$

If $p = \pm 1$, then the condition that each exponent σ_h $(h = 0, 1, \ldots)$ is $\neq 0$ is obviously superfluous, but this is not true if $p \neq \pm 1$. Indeed, if we choose

$$f(y) = (y - p\alpha - q)^{-1}$$

then the left-hand side of (7.6.8) is equal to

$$p \int_{H_{\alpha+}}^{\beta} \frac{dx}{p(x-\alpha)} = \log(\beta - \alpha)$$

and the right-hand side of (7.6.8) is for $p>0$ equal to

$$\int_{H_{p\alpha+q+}}^{p\beta+q} \frac{dy}{y - p\alpha - q} = \log(p\beta - p\alpha) = \log(\beta - \alpha) + \log p$$

so that the right-hand side of (7.6.8) is $\log p$ more than the left-hand side.

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(Received May 2, 1961)

FONCTIONS ENTIÈRES DE p VARIABLES ET FONCTIONS PLURISOUSHARMONIQUES A CROISSANCE TRÈS LENTE

Par

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1. On sait que dans C^1 pour une fonction entière $f(z)$ telle que:

(1) $$\limsup_{r\to\infty} \frac{\operatorname{Log} M(r)}{(\operatorname{Log} r)^2} < \infty$$

$M(r) = \sup_{|z|=r} |f|$, on a si $f(0) \neq 0$

(2) $$\operatorname{Log} M(r) \sim \int_0^r \frac{n(x)}{x}\, dx, \qquad r \to \infty$$

où $n(x)$ désigne le nombre des zeros de $f(z)$ dont le module est inférieur à x. D'une façon générale il existe encore une suite infinie de valeurs indéfiniment croissantes de r telle que (2) ait lieu lorsque $f(z)$ est d'ordre nul mais ne satisfait pas nécessairement à (1) (cf. [1]). D'autre part, il est connu que sous l'hypothèse (1), on a:

(3) $$\operatorname{Log} |f(z)| \sim \operatorname{Log} M(r), \qquad r \to \infty$$

à la condition d'exclure certains cercles contenant les zéros. Les cercles exclus sont vus de l'origine sous des angles qui tendent vers zéro lorsque le centre s'éloigne indéfiniment. Les résultats ci-dessus ont été généralisés récemment par W. K. Hayman aux fonctions sousharmoniques d'une variable complexe (cf. [2]).

Pour passer à $p > 1$ variables nous étudions d'abord les fonctions plurisousharmoniques à croissance lente. Nous démontrons l'égalité asymptotique (2) pour ces fonctions et l'égalité (3) dans le cas des polynômes $P(z_1, \dots, z_p)$ en des points $Z = (z_1, \dots, z_p)$ à distance au moins $\varphi(|z|)$ $\left(\text{avec } \lim_{|z|\to\infty} \frac{\varphi(|Z|)}{|Z|} = 0\right)$ de l'ensemble analytique $P(z_1, \dots, z_p) = 0$.

La démonstration est basée sur une représentation intégrale des fonctions plurisousharmoniques due à P. Lelong (cf. [3]) qui, dans le cas où elles sont d'ordre < 1 s'énonce sous la forme du Théorème I.

Nous utilisons aussi une propriété de l'ensemble W^{p-1} défini par les points d'un ensemble analytique complexe ; cette propriété permet de majorer l'étendue des points de C^p à distance $<\delta$ $(\delta > 0)$ de W^{p-1} et est due également à P. Lelong (cf. [4]).

Notations — Nous renvoyons à [5] pour la définition et les propriétés des fonctions plurisousharmoniques. Si $V(z_1, \dots, z_p)$ est une telle fonction, nous désignons par $\lambda(V ; Z ; r)$ sa moyenne sur la frontière de la boule $B(Z ; r)$ de centre $Z = (z_1, \dots, z_p)$ et de rayon r dans C^p ; $d\mu$ sera la mesure de Radon positive associée à V en tant que fonction sousharmonique dans R^{2p} support de C^p, et $\mu(Z ; r)$ la masse portée par $B(Z ; r)$. Nous posons :

$$\nu(Z ; r) = \frac{\partial\lambda(V ; Z ; r)}{\partial \operatorname{Log} r}, \quad |Z| = \left(\sum_{i=1}^{p} z_i \bar{z}_i\right)^{1/2} ;$$

$$\nu(0 ; r) = \nu(r)$$

(si $\nu(Z ; r)$ est une dérivée à gauche ou à droite, elle sera désignée respectivement par ν_g et ν_d) :

Rappelons que l'on a (cf. 5) :

$$\mu(Z : r) = \frac{1}{2p-2} r^{2p-2} \nu(Z ; r) ; \qquad p \geqq 2 .$$

Si $f(z_1, \dots, z_p)$ est analytique, $d\sigma$ l'élément d'aire de l'ensemble analytique $f = 0$, et $d\mu$ la mesure de Radon associée à $V = \operatorname{Log}|f|$, on a :

$$d\mu = \omega_{2p-2}^{-1}\, d\sigma$$

où ω_{2p-2} est la mesure de la frontière de la boule $B(0 ; 1) \subset C^{p-1}$.

Théorème [1] I (P. Lelong) : *Soit V une fonction plurisousharmonique dans tout C^p, satisfaisant aux conditions :*

a) $\lambda(V^+ ; 0 ; r) = o(r), \quad r \to \infty$

$V^+ = \sup(V, 0)$.

b) $\displaystyle\int_1^{\infty} \frac{d\nu(t)}{t} < \infty ;$

1. On trouvera dans [6] une démonstration directe. La démonstration donnée dans [3] pour les polynômes s'applique d'ailleurs au cas de l'ordre inférieur à 1.

alors si $V(0) > -\infty$, on a:

$$V(z_1, \dots, z_p) = V(0) + \int_{C^p} d\mu(\xi)\left[\frac{1}{|\xi|^{2p-2}} - \frac{1}{|\xi - Z|^{2p-2}}\right].$$

2. Démontrons d'abord quelques lemmes.

Lemme 1 — Soit $\lambda(t)$ une fonction convexe croissante de $\operatorname{Log} t$ sur la demi-droite $t>0$, telle que pour $t>t_0$ on ait:

$$0 < A \leqq \frac{\lambda(t)}{(\operatorname{Log} t)^m} \leqq B < \infty \qquad m \geqq 1$$

où A et B désignent deux constantes ne dépendant que de t_0. Alors pour toute suite $t_n \to \infty$, telle que $\operatorname{Log} t_{n+1} \sim \operatorname{Log} t_n$, on aura:

$$\lim_{n\to\infty} \frac{\lambda(t_n)}{\lambda(t_{n+1})} = 1 .$$

En effet, si ν_d est la dérivée à droite de $\lambda(t)$ par rapport à $\operatorname{Log} t$, on a:

$$\lambda(t_{n+1}) - \lambda(t_n) = \int_{t_n}^{t_{n+1}} \nu_d(t) \operatorname{Log} t \leqq \nu_d(t_{n+1}) (\operatorname{Log} t_{n+1} - \operatorname{Log} t_n)$$

donc, si $t_n > \operatorname{Sup}(1, t_0)$,

$$(4) \qquad 1 - \frac{\lambda(t_n)}{\lambda(t_{n+1})} \leqq \frac{\nu_d(t_{n+1}) \operatorname{Log} t_{n+1}}{\lambda(t_{n+1})}\left[1 - \frac{\operatorname{Log} t_n}{\operatorname{Log} t_{n+1}}\right]$$

et :

$$\nu_d(t) \leqq \frac{\lambda(t^2) - \lambda(t)}{\operatorname{Log} t} \leqq \frac{B[\operatorname{Log}(t^2)]^m}{\operatorname{Log} t} = 2^m B (\operatorname{Log} t)^{m-1}$$

$$(t > \operatorname{Sup}(1, t_0)).$$

Le premier facteur du second membre de (4) est pour n assez grand borné par $\dfrac{2^m B (\operatorname{Log} t_{n+1})^{m-1}}{\lambda(t_{n+1})}$ qui reste borné supérieurement lorsque $n \to \infty$.

Comme le premier membre de (4) est positif et le crochet tend vers zéro, il en résulte le lemme.

Lemme 2 — Soit $\lambda(t)$ une fonction convexe (non constante) croissante de $\operatorname{Log} t$ sur la demi-droite $t > 0$. Si :

$$\lambda(t) = O(\operatorname{Log} t), \qquad t \to \infty$$

alors :

$$\lim_{n \to \infty} \frac{\nu_d(t_{n+1})}{\nu_g(t_n)} = 1$$

pour toute suite croissante $t_n \to \infty$, telle que $\operatorname{Log} t_n \sim \operatorname{Log} t_{n+1}$.

En effet, ν_g et ν_d tendent en croissant vers $\lim\limits_{t \to \infty} \dfrac{\lambda(t)}{\operatorname{Log} t}$ qui existe et est finie $\neq 0$, donc :

$$\nu_d(t) = \frac{\lambda(t)}{\operatorname{Log} t} + \varepsilon_1(t) \qquad (\varepsilon_1 \to 0)$$

$$\nu_g(t) = \frac{\lambda(t)}{\operatorname{Log} t} + \varepsilon_2(t) \qquad (\varepsilon_2 \to 0)$$

et :

$$(5) \qquad \frac{\nu_d(t_{n+1})}{\nu_g(t_n)} \cdot \frac{\operatorname{Log} t_{n+1}}{\operatorname{Log} t_n} = \frac{\lambda(t_{n+1}) + \varepsilon_1(t_{n+1}) \operatorname{Log} t_{n+1}}{\lambda(t_n) + \varepsilon_2(t_n) \log t_n}$$

$$= \frac{1 + \varepsilon_1(t_{n+1})\, \lambda^{-1}(t_{n+1}) \operatorname{Log} t_{n+1}}{\dfrac{\lambda(t_n)}{\lambda(t_{n+1})} + \varepsilon_2(t_n)\, \lambda^{-1}(t_{n+1}) \operatorname{Log} t_n} .$$

Comme $\lambda^{-1}(t_{n+1}) \operatorname{Log} t_{n+1}$, $\lambda^{-1}(t_{n+1}) \operatorname{Log} t_n$ restent bornées et $\dfrac{\lambda(t_n)}{\lambda(t_{n+1})} \to 1$ (Lemme 1), $\dfrac{\operatorname{Log} t_{n+1}}{\operatorname{Log} t_n} \to 1$, on déduit de (5) le Lemme 2.

Proposition 1 — Soit $V(z_1, \ldots, z_p)$ une fonction plurisousharmonique (non constante) dans tout C^p telle que :

$$\lambda(V ; 0 ; r) = O(\operatorname{Log} r), \qquad r \to \infty$$

si R_n est une suite croissante ($R_n \to \infty$) telle que $R_n \sim R_{n+1}$, alors,

$$\lim_{n \to \infty} \frac{\mu(R_n ; R_{n+1})}{\mu(R_n)} = 0 ,$$

où $\mu(R_n, R_{n+1})$ est la masse de V portée par l'ensemble $R_n \leqq |Z| \leqq R_{n+1}$ et $\mu(R_n)$ celle portée par la boule $B(0 ; R_n)$.

En effet :

$$\begin{aligned}(6)\qquad \frac{\mu(R_n, R_{n+1})}{\mu(R_n)} &= \frac{1}{R_n^{2p-1}\, \nu_g(R_n)} \left[R_{n+1}^{2p-2}\, \nu_d(R_{n+1}) - R_n^{2p-2}\, \nu_g(R_n)\right] \\ &= \left(\frac{R_{n+1}}{R_n}\right)^{2p-2} \frac{\nu_d(R_{n+1})}{\nu_g(R_n)} - 1 .\end{aligned}$$

Sous l'hypothèse $R_n \sim R_{n+1}$ le Lemme 2 s'applique. Le troisième membre de (6) tend vers zéro.

Définition — Une fonction plurisousharmonique $V(z_1, \ldots, z_p)$ dans tout C^p sera dite d'ordre α si :

$$\limsup_{r\to\infty} \frac{\operatorname{Log} M(r)}{\operatorname{Log} r} = \alpha$$

$$M(r) = \sup_{|z|\leqq r} V(z_1, \ldots, z_p) .$$

Une fonction entière $f(z_1, \ldots, z_p)$ sera dite d'ordre α si la fonction $V = \operatorname{Log}|f|$ est d'ordre α.

Lemme 3 — Pour une fonction plurisousharmonique d'ordre nul, on a :

$$\liminf_{r\to\infty} \frac{\nu(r)}{\lambda(V ; 0 ; r)} = 0 .$$

En effet, supposons le contraire, il existe alors un nombre positif h tel que :

$$\frac{\nu(r)}{\lambda(V ; 0 ; r)} = \frac{\dfrac{d\lambda(V ; 0 ; r)}{d \operatorname{Log} r}}{\lambda(V ; 0 ; r)} \geqq h > 0$$

ou : $\lambda(V ; 0 ; r) > k\, r^h$ (k étant une constante > 0) ; V ne sera pas d'ordre nul.

3. Proposition 2 — Soit $V(z_1, \ldots, z_p)$ une fonction plurisousharmonique dans tout C^p (non constante) telle que :

$$(7)\qquad \limsup_{r\to\infty} \frac{M(r)}{(\operatorname{Log} r)^m} < \infty , \qquad m \geqq 1 .$$

On a, si $V(0) > -\infty$:

$$(8) \qquad M(r) = \int_0^r \frac{\nu(t)}{t}\,dt + (2p-1)\,\vartheta_1(r)\,r \int_r^\infty \frac{\nu(t)}{t^2}\,dt - \varphi_1(r)$$

avec $0 < \vartheta_1(r) < 1$; $\varphi_1(r) > 0$; et $\limsup_{r\to\infty} \dfrac{\varphi_1(r)}{(\operatorname{Log} r)^{m-1}} < \infty$, si $\nu_0 = \lim_{t\to\infty} \nu(t) = \infty$; $\varphi_1(r) \leqq \nu_0$, si $\nu_0 < \infty$.

D'une façon générale, pour une fonction plurisousharmonique $V(z_1, \ldots, z_p)$ $(V(0) > -\infty)$ d'ordre nul, on a[2].

$$(9) \qquad M(r) = \int_0^r \frac{\nu(t)}{t}\,dt + (2p-1)\,\vartheta_2(r)\,r \int_r^\infty \frac{\nu(t)}{t^2}\,dt - \varphi_2(r)$$

avec: $0 < \vartheta_2(r) < 1$, $\varphi_2(r) > 0$, et :

$$\liminf_{r\to\infty} \frac{\varphi_2(r)}{\displaystyle\int_0^r \frac{\nu(t)}{t}\,dt} = 0 .$$

Une fonction plurisousharmonique d'ordre nul vérifie les hypothèses du Théorème I si $V(0) > -\infty$. En effet, on a pour t assez grand $\nu(t) < Bt^\varepsilon$, $\varepsilon > 0$ arbitrairement petit. D'où la condition (b) du Théorème I. On a donc en supposant pour commodité d'écriture $V(0) = 0$, et sous réserve de la convergence de l'intégrale :

$$(10) \qquad V(z_1, \ldots, z_p) \leqq \int_{C^p} d\mu(\xi) \left[\frac{1}{|\xi|^{2p-2}} - \frac{1}{(|\xi| + |Z|)^{2p-2}} \right].$$

Posons $|\xi| = t$, $|Z| = r$, et remarquons que l'on a :

$$\lim_{t\to 0} \frac{\mu(t)}{t^{2p-2}} = 0$$

$$\frac{1}{t^{2p-2}} - \frac{1}{(t+r)^{2p-2}} \leqq \frac{2p-2}{t^{2p-1}}\,r \qquad (t + r \neq 0)$$

$$\lim_{t\to\infty} (2p-2)\,\frac{\mu(t)}{t^{2p-1}} = \lim_{t\to\infty} \frac{\nu(t)}{t} = 0 .$$

2. L'égalité (9) est à rapprocher d'un résultat de P. Lelong (cf. [3]).

L'intégrale figurant au second membre de (10) s'écrit grâce à une intégration par parties :

(11)
$$\int_0^\infty \nu(t)\left[\frac{1}{t} - \frac{t^{2p-2}}{(t+r)^{2p-1}}\right]dt = \int_0^r \frac{\nu(t)}{t}\,dt$$
$$+\int_r^\infty \frac{\nu(t)}{t}\left[1 - \frac{1}{(1+\frac{r}{t})^{2p-1}}\right]dt - \int_0^r \frac{t^{2p-2}}{(t+r)^{2p-1}}\,\nu(t)\,dt\,.$$

Or pour $x = \frac{r}{t} < 1$, on a :

$$1-(1+x)^{-(2p-1)} < (2p-1)\,x$$

donc la deuxième intégrale du second membre de (11) est majorée par

(12)
$$(2p-1)\,r\int_r^\infty \frac{\nu(t)}{t^2}\,dt < \infty$$

et la dernière intégrale est une fonction positive $\varphi(r)$ telle que :

(13)
$$\varphi(r) = \int_0^1 \frac{u^{2p-2}}{(u+1)^{2p-1}}\,\nu(ru) \leqq \nu(r)\,.$$

D'après (10), (11) et (12) on obtient :

$$\int_0^r \frac{\nu(t)}{t}\,dt \leqq M(r) \leqq \int_0^r \frac{\nu(t)}{t}\,dt + (2p-1)\,r\int_r^\infty \frac{\nu(t)}{t^2}\,dt - \varphi(r)\,,$$

où

$$M(r) = \int_0^r \frac{\nu(t)}{t}\,dt + (2p-1)\,\vartheta(r)\,r\int_r^\infty \frac{\nu(t)}{t^2}\,dt - \vartheta(r)\,\varphi(r)$$

avec $0<\vartheta(r)<1$. Dans le cas (7) et suivant que $\nu_0 = \lim\limits_{t\to\infty} \nu(t)$ est finie ou infinie, on a : $\varphi(r) \leqq \nu_0$ et $\varphi(r) \leqq B(\mathrm{Log}\, r)^{m-1}$ $(r>r_0)$ (d'après (13)).

Dans tous les cas on a (Lemme 3)

$$\liminf_{r\to\infty} \frac{\varphi(r)}{\displaystyle\int_0^r \frac{\nu(t)}{t}\,dt} = 0\,,$$

ce qui établit la Proposition 2.

Théorème 1 — Pour une fonction plurisousharmonique $V(z_1, \ldots, z_p)$ $(V(0) > -\infty)$ non constante, vérifiant :

$$M(r) = O\left[(\operatorname{Log} r)^2\right], \qquad r \to \infty$$

on a :

$$(14) \qquad M(r) \sim \int_0^r \frac{\nu(t)}{t}\, dt \sim \lambda(V\,;0\,;r), \qquad r \to \infty\,.$$

D'une façon générale pour une fonction plurisousharmonique $V(z_1, \ldots, z_p)$ d'ordre nul $(V(0) > -\infty)$, il existe encore une suite de valeurs de r tendant vers l'infini pour laquelle (14) est vérifié.

En effet, si $\lim\limits_{t \to \infty} \nu(t) = \nu_0 = \infty$, on a pour $r > r_0$,

$$(15) \quad r \int_r^\infty \frac{\nu(t)}{t^2}\, dt \leqq Br \int_r^\infty \frac{\operatorname{Log} t}{t^2}\, dt = B(\operatorname{Log} r + 1) = o\left(\int_0^r \frac{\nu(t)}{t}\, dt\right), \quad r \to \infty\,.$$

Si $\nu_0 < \infty$, et comme $\lambda(V\,;0\,;r)$ croît indéfiniment,

$$(15') \qquad r \int_r^\infty \frac{\nu(t)}{t^2}\, dt < \nu_0 = o\left(\int_0^r \frac{\nu(t)}{t^2}\, dt\right), \qquad r \to \infty\,.$$

Dans tous les cas, on aura :

$$(16) \qquad \liminf_{r \to \infty} \frac{r \displaystyle\int_r^\infty \frac{\nu(t)}{t^2}\, dt}{\displaystyle\int_0^r \frac{\nu(t)}{t}\, dt} = 0\,.$$

En effet, en reprenant un raisonnement déjà fait dans le cas des fonctions entières d'une variable complexe (Voir par exemple [1]), on aura si (16) n'est pas vérifiée,

$$r \int_r^\infty \frac{\nu(t)}{t^2}\, dt > \alpha \int_0^r \frac{\nu(t)}{t}\, dt \qquad (\alpha > 0\,,\, r > r_0)\,.$$

En supposant (sans restreindre la généralité) $V(0) = 0$, il vient alors :

$$\alpha\lambda(V\,;\,0\,;\,r) < r \int_r^\infty \frac{d\lambda(V\,;\,0\,;\,r)}{d \operatorname{Log} t} \frac{d \operatorname{Log} t}{t}.$$

D'où par une intégration par parties :

$$(1+\alpha)\frac{d}{d \operatorname{Log} t}\int_r^\infty \frac{\lambda(V\,;\,0\,;\,t)}{t}\, d\operatorname{Log} t + \int_r^\infty \frac{\lambda(V\,;\,0\,;\,t)}{t}\, d\operatorname{Log} t > 0.$$

Par suite :

$$-\frac{\dfrac{d}{dr}\displaystyle\int_r^\infty \frac{\lambda(V\,;\,0\,;\,t)}{t^2}\,dt}{\displaystyle\int_r^\infty \frac{\lambda(V\,;\,0\,;\,t)}{t^2}\,dt} + \frac{1}{(1+\alpha)r} = \frac{d}{dr}\operatorname{Log}\left[r^{\frac{1}{1+\alpha}}\int_r^\infty \frac{\lambda(V\,;\,0\,;\,t)}{t^2}\,dt\right] > 0$$

ou

$$r\int_r^\infty \frac{\lambda(V\,;\,0\,;\,t)}{t^2}\,dt > k\, r^{\frac{\alpha}{\alpha+1}} \qquad (k = \text{const.}).$$

Il en résulte que pour une certaine suite $r_n \to \infty$, $\lambda(V\,;\,0\,;\,r_n)\, r_n^{-\alpha/(\alpha+1)}$ ne tend pas vers zéro. Donc :

$$M(r_n) > \lambda(V\,;\,0\,;\,r_n) > k_1\, r_n^{\frac{\alpha}{\alpha+1}} \qquad (k_1 = \text{const.})$$

c'est à dire V ne serait pas d'ordre nul. Le théorème résulte alors de la Proposition 2 et de l'égalité (9). En effet, on aura :

$$(17) \qquad \frac{M(r)}{\displaystyle\int_0^r \frac{\nu(t)}{t}\,dt} = 1 + (2p-1)\,\vartheta_2(r)\frac{r\displaystyle\int_r^\infty \frac{\nu(t)}{t^2}\,dt}{\displaystyle\int_0^r \frac{\nu(t)}{t}\,dt} - \frac{\varphi_2(r)}{\displaystyle\int_0^r \frac{\nu(t)}{t}\,dt}.$$

Dans tous les cas le dernier terme de (17) tend vers zéro (d'après (13)). Le second terme du deuxième membre de (17) tend également vers zéro lorsque $r \to \infty$ si on est dans le cas où $M(r) = O[(\operatorname{Log} r)^2]$ (d'après (15)

et (15')). Sinon il tend vers zéro pour une certaine suite r_n (d'après (16)). D'où le théorème.

4. Dans [6] nous avons établi pour $m = 1$ l'énoncé suivant :

Théorème 2 — Soit $V(z_1, \dots, z_p)$ une fonction plurisous-harmonique dans tout C^p. Si l'on a:

$$\lambda(V^+ ; 0 ; r) = O[\operatorname{Log} r)^m], \qquad m \geqq 1 \ , \ r \to \infty ,$$

alors,

$$\limsup_{r\to\infty} \frac{\lambda(V^+ ; P ; r)}{(\operatorname{Log} r)^m} = \limsup_{r\to\infty} \frac{M_P(r)}{(\operatorname{Log} r)^m}$$

où $M_P(r) = \operatorname{Sup} V$ dans $B(P ; r)$. La valeur commune étant indépendante du point $P \in C^p$.

Le cas $m > 1$ peut être démontré de la même manière que dans [6]. En effet, supposons tout d'abord le point P à l'origine. Soit $B(0 ; r) \subset B(0 ; R)$ et posons $k = \dfrac{r}{R}$. En écrivant que les valeurs prises par V sur la frontière de la boule $B(0 ; r)$ sont au plus égales à celles prises par la fonction harmonique qui coïncide avec V sur la frontière de $B(0 ; R)$, on obtient dans $B(0 ; r)$ grâce à l'intégrale de Poisson :

$$(18) \quad \lambda(V^+ ; 0 ; r) \leqq M(r) \leqq \frac{1}{\omega_{2p-1}(1)} \int V(Q) \frac{R^2 - r^2}{|Z-Q|^{2p}} R^{2p-2}\, d\omega_{2p-1}(\vec{a})$$
$$\leqq \frac{1+k}{(1-k)^{2p-1}} \lambda(V^+ ; 0 ; R)$$

$Q = R\vec{a}, \quad |Z| = r, \quad |\vec{a}| = 1 .$

Soit $R = R(r) = r \operatorname{Log} r \quad (r > 1)$. On a:

$$\lim_{r\to\infty} k(r) = 0, \quad \lim_{r\to\infty} \frac{\operatorname{Log} r}{\operatorname{Log} R(r)} = 1 ,$$

donc, d'après (18)

$$\chi(0) = \limsup_{r\to\infty} \frac{\lambda(V^+ ; 0 ; r)}{(\operatorname{Log} r)^m} \leqq \limsup_{r\to\infty} \frac{M(r)}{(\operatorname{Log} r)^m}$$
$$\leqq \lim_{r\to r} \left[\frac{1+k}{(1-k)^{2p-1}} \left(\frac{\operatorname{Log}(r \operatorname{Log} r)}{\operatorname{Log} r}\right)^m\right] \limsup_{r\to\infty} \frac{\lambda(V^+ ; 0 ; r \operatorname{Log} r)}{[\operatorname{Log}(r \operatorname{Log} r)]^m} \leqq \chi(0) .$$

D'où le théorème dans le cas où P est à l'origine. Supposons maintenant P différent de 0. Si $|OP| = \rho$, on aura $M_P(r) \leq M(r+\rho)$, donc:

(19)

$$\chi(P) = \limsup_{r\to\infty} \frac{M_P(r)}{(\operatorname{Log} r)^m} \leq \limsup_{r\to\infty} \frac{M(r+\rho)}{(\operatorname{Log}(r+\rho))^m} \lim_{r\to\infty} \left(\frac{\operatorname{Log}(r+\rho)}{\operatorname{Log} r}\right)^m \leq \chi(0).$$

De même, $M(r) \leq M_P(r+\rho)$, donc:

(20)

$$\chi(0) = \limsup_{r\to\infty} \frac{M(r)}{(\operatorname{Log} r)^m} \leq \limsup_{r\to\infty} \frac{M_P(r+\rho)}{(\operatorname{Log}(r+\rho))^m} \lim_{r\to\infty} \left(\frac{\operatorname{Log}(r+\rho)}{\operatorname{Log} r}\right)^m \leq \chi(P).$$

D'où le théorème d'après (19) et (20).

Théorème 3 — Pour une fonction plurisousharmonique $V(z_1, \dots, z_p)$ dans tout C^p (non constante) avec $V(0) > -\infty$, et telle que:

$$M(r) = O[(\operatorname{Log} r)^2], \qquad r \to \infty,$$

on a:

$$\liminf_{r\to\infty} \frac{\lambda(V^-; 0; r)}{(\operatorname{Log} r)^2} = 0;$$

$V^- = \operatorname{Sup}(-V, 0)$. Si $M(r) = O[(\operatorname{Log} r)]$, on a:

$$\lim_{r\to\infty} \frac{\lambda(V^-; 0; r)}{\operatorname{Log} r} = 0.$$

En effet, d'après le Théorème 2, $M(r) \sim \lambda(V; 0; r)$, $r \to \infty$, donc

(21)
$$\begin{aligned}
\limsup_{r\to\infty} \frac{M(r)}{(\operatorname{Log} r)^2} &= \limsup_{r\to\infty} \frac{\lambda(V; 0; r)}{(\operatorname{Log} r)^2} \\
&= \limsup_{r\to\infty} \left[\frac{\lambda(V^+; 0; r)}{(\operatorname{Log} r)^2} - \frac{\lambda(V^-; 0; r)}{(\operatorname{Log} r)^2}\right] \\
&\leq \limsup_{r\to\infty} \frac{\lambda(V^+; 0; r)}{(\operatorname{Log} r)^2} - \liminf_{r\to\infty} \frac{\lambda(V^-; 0; r)}{(\operatorname{Log} r)^2}.
\end{aligned}$$

Donc, d'après le Théorème 2 et (21),

$$0 \leq -\liminf_{r\to\infty} \frac{\lambda(V^-; 0; r)}{(\operatorname{Log} r)^2}.$$

D'où la conclusion. Si $M(r) = O(\operatorname{Log} r)$, on a (Th. 2):

$$\lim_{r\to\infty} \frac{M(r)}{\operatorname{Log} r} = \lim_{r\to\infty} \frac{\lambda(V^+ ; 0 ; r)}{\operatorname{Log} r} - \lim_{r\to\infty} \frac{\lambda(V^- ; 0 ; r)}{\operatorname{Log} r}$$

car les deux premières limites existent. D'où la conclusion d'après le Théorème 3.

Corollaire — Si $\lim\limits_{r\to\infty} \dfrac{M(r)}{(\operatorname{Log} r)^2}$ existe, alors

$$\lim_{r\to\infty} \frac{\lambda(V^- ; 0 ; r)}{(\operatorname{Log} r)^2} = 0 .$$

5. Dans la suite nous désignons par W_f^{p-1} l'ensemble analytique défini par $f(Z_1, \dots, Z_p) = 0$ où f est une fonction entière ; par

$$W_{f,t}^{p-1} = [W_f^{p-1} \cap (|Z| < t)]$$

la restriction de W_f^{p-1} à la boule $B(0 ; t)$. Nous écrivons $Z = r\vec{a}$ où $\vec{a}$ est un vecteur unitaire de C^p. La distance d'un point Z à un ensemble de points $E \subset C^p$ sera notée $D(Z ; E)$:

$$D(Z ; E) = \inf_{\xi \in E} |Z - \xi| .$$

Théorème 4 — Soit $f(Z_1, \dots, Z_p)$ une fonction entière (non constante) d'ordre nul avec $f(0) = 1$; soit $R > D(0 ; W_f^{p-1})$. Si $\tau > 1$ est un nombre vérifiant :

$$(\tau - 1) R < \sup_{|Z| < R} D(Z ; W_f^{p-1})$$

alors il existe une constante $k(p, \tau)$ telle que pour $|Z| \leq R$ et à l'extérieur de N boules de rayon $\rho = (\tau - 1) R$ centrées sur $W_{f,\tau R}^{p-1}$, on ait :

$$(22) \qquad \operatorname{Log} |f(Z_1, \dots, Z_p)| \geq M(\tau R) - k(p ; \tau) R \int_{\tau R}^{\infty} \frac{\nu(t)}{t^2} dt$$

où

$$\nu(t) = \frac{d\lambda(\operatorname{Log} |f| ; 0 ; t)}{d \operatorname{Log} t} ,$$

et

$$M(\tau R) = \sup_{\vec{a}} \operatorname{Log} |f(\tau R \vec{a})| .$$

On a

$$k(p\,,\tau) \leq k_1(\alpha\,,p)\,\frac{\tau}{(\tau-1)^{2p}}$$

où α est un nombre quelconque qui majore τ; k_1 ne dépend que de α et p. Le nombre N est inférieur à

$$k_2(p)\left(\frac{3\tau-2}{\tau-1}\right)^{2p-2} \nu\,[(3\tau-2)R]\,.$$

k_2 étant une constante ne dépendant que de p.

Rappelons tout d'abord un théorème de P. Lelong (cf. [4]) qui s'énonce :

Théorème II — Soit W^{p-1} un ensemble analytique complexe défini localement sur un domaine D fermé, d'aire totale $\mu(D)$ sur D. Si η_ρ est un ensemble de points de D tels que:

a) Ils sont à distance au plus $\rho>0$ de W^{p-1}

b) Ils sont points intérieurs de D, à distance au moins

$$\delta = 3\rho\sqrt{2p} \qquad \text{de la frontière.}$$

Alors η_ρ peut être recouvert par N boules égales centrées sur W^{p-1} de rayon $2\rho\sqrt{2p}$ pour lequel on a $N\rho^{2p-2} \leqq K\mu(D)$ où K ne dépend que de p.

Ce théorème étant rappelé, soit η l'ensemble des points de C^p qui sont à distance $(\tau-1)\dfrac{R}{2\sqrt{2p}} = \rho_1$ au plus de la partie de W_f^{p-1} contenue dans $B(0\,;\tau R)$. Les points de η sont les points intérieurs de la boule $B[0\,;(3\tau-2)R]$ à distance au moins $3\rho_1\sqrt{2p}$ de sa frontière. Donc, d'après l'énoncé ci-dessus on péut recouvrir η par N boules centrées sur $W^{p-1}_{f,(3\tau-2)R}$ de rayon $(\tau-1)R=\rho$, avec

$$N\rho^{2p-2} \leqq K(p)\,\mu[\,3\tau-2)R] \qquad (K(p) = \text{const.}))$$

où

$$N \leqq k_2(p)\left(\frac{3\tau-2}{\tau-1}\right)^{2p-1} \nu\,[(3\tau-2)R]\,.$$

Remarquons que seules les boules centrées sur $W^{p-1}_{f,\tau R}$ peuvent rencontrer la boule $B(0\,;R)$. Soit Z un point de $B(0\,;R)$ extérieur à N boules de recouvrement ci-dessus. D'après le Théorème I, on a :

$$\operatorname{Log}|f(z_1,\dots,z_p)| = \int_{|\xi|<\tau R} d\mu(\xi)\left[\frac{1}{|\xi|^{2p-2}} - \frac{1}{|\xi-Z|^{2p-2}}\right]$$

$$+ \int_{|\xi|\geqq\tau R} d\mu(\xi)\left[\frac{1}{|\xi|^{2p-2}} - \frac{1}{|\xi-Z|^{2p-2}}\right] = I_1 + I_2 .$$

Minorons l'intégrale I_1. Comme $|\xi-Z| > \rho = (\tau-1)R$,

$$I_1 \geqq \int_{|\xi|<\tau R} \frac{d\mu(\xi)}{|\xi|^{2p-2}} - \frac{1}{\rho^{2p-2}}\,\mu(\tau R)\,;$$

une intégration par parties donne alors :

$$(23)\quad I_1 \geqq \frac{1}{2p-2}\,\nu(\tau R) + \int_0^{\tau R} \frac{\nu(t)}{t}\,dt - \frac{1}{2p-2}\left(\frac{\tau}{\tau-1}\right)^{2p-2}\nu(\tau R)\,;$$

or

$$(24)\qquad r\int_r^{\infty}\frac{\nu(t)}{t^2}\,dt > r\,\nu(r)\int_r^{\infty}\frac{dt}{t^2} = \nu(r).$$

Donc d'après (23) et (24),

$$(25)\qquad I_1 \geqq \int_0^{\tau R}\frac{\nu(t)}{t}\,dt - \frac{1}{2p-2}\left(\frac{1}{\tau-1}\right)^{2p-2}\left(\tau R\int_{\tau R}^{\infty}\frac{\nu(t)}{t^2}\,dt\right).$$

D'autre part, si $|\xi|\geqq\tau R$, on aura $|\xi-Z| > |\xi|-r$ $(r=|Z|)$ et ;

$$\frac{1}{|\xi|^{2p-2}} - \frac{1}{|\xi-Z|^{2p-2}} \geqq \frac{1}{|\xi|^{2p-2}} - \frac{1}{(|\xi|-r)^{2p-2}}$$

$$\geqq -(2p-2)\frac{r}{(|\xi|-r)^{2p-1}}.$$

Donc :

$$I_2 \geqq -(2p-2)\,r\int_{|\xi|>\tau R}\frac{d\mu(\xi)}{(|\xi|-r)^{2p-1}}$$

ou grâce à une intégration par parties,

$$I_2 \geqq -(2p-1)\, r \left(1 - \frac{r}{\tau R}\right)^{-2p} \int_{\tau R}^{\infty} \frac{\nu(t)}{t^2}\, dt$$

et en remarquant que l'on a; $r < R$, $\tau > 1$ et (24),

$$I_2 \geqq -(2p-1)\left(\frac{\tau}{\tau-1}\right)^{2p} \left(\tau R \int_{\tau R}^{\infty} \frac{\nu(t)}{t^2}\, dt\right). \tag{26}$$

Par conséquent, d'après (25) et (26), on obtient :

$$\operatorname{Log} |f(z_1, \ldots, z_p)| \geqq I_1 + I_2 \geqq \int_0^{\tau R} \frac{\nu(t)}{t}\, dt \tag{27}$$

$$-\left[\frac{1}{2p-2}\left(\frac{\tau}{\tau-1}\right)^{2p-2} + (2p-1)\left(\frac{\tau}{\tau-1}\right)^{2p}\right] \tau R \int_{\tau R}^{\infty} \frac{\nu(t)}{t^2}\, dt .$$

D'après la Proposition 2, on a :

$$\int_0^{\tau R} \frac{\nu(t)}{t}\, dt \geqq M(\tau R) - (2p-1)\left(\tau R \int_{\tau R}^{\infty} \frac{\nu(t)}{t^2}\, dt\right). \tag{28}$$

Donc, d'après (27) et (28),

$$\operatorname{Log} |f(z_1, \ldots, z_p)| \tag{29}$$

$$\geqq M(\tau R) - \frac{\tau}{(\tau-1)^{2p}}\left[\frac{1}{2p-2}\, \tau^{2p-2} (\tau-1)^2 + (2p-1)\, \tau^{2p}\right.$$

$$\left. + (2p-1)(\tau-1)^{2p}\right] R \int_{\tau R}^{\infty} \frac{\nu(t)}{t^2}\, dt .$$

Le crochet figurant au second membre de (29) est borné par un nombre $K_1(\alpha, p)$ qui ne dépend que de p et $\alpha > \tau$. D'où le Théorème 4.

Proposition 3 — Soit $P(Z_1, \ldots, Z_p)$ un polynôme de degré $q > 0$ par rapport à l'ensemble des variables. Soit

$$\delta(r) = \sup_{|Z|=r} D(Z\,;\, W_P^{p-1}),$$

alors

$$a(p\,,q) = \liminf_{r\to\infty} \frac{\delta(r)}{r} > 0\,.$$

En effet, supposons le contraire; il y a alors une suite croissante de nombres $r_n \to \infty$, avec

$$\lim_{n\to\infty} \frac{\delta(r_n)}{r_n} = 0\,.$$

Soit

$$\varepsilon_n = \frac{\delta(r_n)}{r_n}\,.$$

Quel que soit le point Z, $|Z| = r_n$, on aura $B(Z\,;\varepsilon'_n r_n) \cap W_P^{p-1} \neq \emptyset$ pour $\varepsilon'_n > \varepsilon_n$. Considérons les domaines

$$A_n \qquad (1-\varepsilon'_n)\,r_n < |Z| < (1+\varepsilon'_n)\,r_n$$

$$A'_n \qquad (1-7\sqrt{2p}\,\varepsilon'_n)\,r_n < |Z| < (1+7\sqrt{2p}\,\varepsilon'_n)\,r_n$$

pour $n > n_0$, n_0 assez grand pour que $7\sqrt{2p}\,\varepsilon'_n < 1$.

La distance d'un point $Z \in A_n$ à l'ensemble $A'_n \cap W_P^{p-1}$ est au plus $2\varepsilon'_n r_n$. Les points de A_n sont d'autre part à une distance au moins $6\sqrt{2p}\,\varepsilon' r_n$ de la frontière de A'_n. Donc, si $\mu(A'_n)$ est la masse de $\mathrm{Log}\,|P(Z_1, \ldots, Z_p)|$ portée par A'_n (compact) on peut recouvrir A_n par N boules de rayon $4\sqrt{2p}\,\varepsilon'_n r_n$ (Th. II) avec:

$$N(\varepsilon'_n r_n)^{2p-2} \leqq K(p)\,\mu(A'_n) \leqq K(p)\,\mu\,[0\,;(1+7\sqrt{2p}\,\varepsilon'_n)\,r_n]$$

($K(p) = \text{const.}$). Donc, en rappelant que $\nu(Z\,;t)$ (correspondant à $\mathrm{Log}\,|P|$) est au plus égal a q quel que soit Z et t, on obtient:

$$N \leqq K_1(p\,,q)\,(1+7\sqrt{2p}\,\varepsilon'_n)^{2p-2}\,\varepsilon'^{\,2-2p}_n \qquad n > n_0$$

($K_1 = \text{const.}$).

Écrivons que le volume de A_n est au plus égal à la somme des volumes de N boules de recouvrement ci-dessus. On obtient:

$$(1+\varepsilon'_n)^{2p} - (1-\varepsilon'_n)^{2p} \leqq K_2(p\,,q)\,(1+7\sqrt{2p}\,\varepsilon'_n)^{2p-2}\,\varepsilon'^{\,2}_n \qquad n > n_0$$

($K_2 = \text{const.}$).

Cette dernière inégalité est en contradiction avec le fait que ε'_n puisse être choisie de manière qu'elle tende vers zéro avec $1/n$ (les deux membres de l'inégalité sont alors des infiniments petits par rapport à ε'_n respectivement d'ordre 1 et 2). D'où la Proposition 3.

Théorème 5 — Soit $P(Z_1, \ldots, Z_p)$ $(P(0) = 1)$, un polynôme de degré $q > 0$ par rapport à l'ensemble des variables. Alors pour toute fonction $\varphi(t)$ continue sur $]0, \infty[$, $\varphi(t)$ décroissante vers zéro quand $t \to \infty$, et vérifiant:

$$\lim_{r \to \infty} [\varphi(r)]^{2p} M(r) = \infty \tag{31}$$

où $M(r) = \sup_{\vec{a}} \log |P(r\vec{a})|$, on a uniformément:

$$\operatorname{Log} P(Z_1, \ldots, Z_p) \sim M(r) \qquad r \to \infty$$

hors de l'ensemble E_φ défini par:

$$D(Z\,; W_P^{p-1}) \leqq \varphi(|Z|)\,|Z|\,.$$

Soit $\varphi(t)$ une fonction continue sur $]0, \infty[$, $\varphi(t)$ décroissante vers zéro quand $t \to \infty$. L'ensemble $C^p - E_\varphi$ n'est pas vide; il contient des points Z dont la distance à l'origine est arbitrairement grande (cela résulte immédiatement de la Proposition 3).

Soit $Z \notin E_\varphi$, Z est alors extérieur à toute boule centrée sur W_P^{p-1} de rayon $\varphi(|Z|)\,|Z|$. Appliquons l'inégalité (22) du Théorème 4 au point Z, avec $R = r = |Z|$, $\tau = 1 + \varphi(r)$:

$$\operatorname{Log} |P(Z_1, \ldots, Z_p)| \geqq M(\tau r) - \frac{K_1(\alpha, p)}{[\varphi(r)]^{2p}}\,\tau r \int_{\tau r}^{\infty} \frac{\nu(t)}{t^2}\,dt\,, \tag{32}$$

$Z \notin E_\varphi$, $r \geqq r_0 > 0$.

(Rappelons que α est une constante quelconque qui majore τ (pour $r \geqq r_0$); K_1 ne dépend que de α et p). Or $M(\tau r) > M(r)$ et pour un polynôme de degré q on a $\nu(t) \leqq q$, donc, d'après (32):

$$1 \geqq \frac{\operatorname{Log} P(r\vec{a})}{M(r)} \geqq 1 - \frac{qK_1(\alpha, p)}{[\varphi(r)]^{2p} M(r)} \tag{33}$$

$r\vec{a} \notin E_\varphi$, $r \geqq r_0 > 0$.

Sous l'hypothèse (31) le dernier terme de (33) tend vers zèro ; donc, on a uniformément :

$$\lim_{r\to\infty} \frac{\operatorname{Log}|P(r\vec{a})|}{M(r)} = 1 \qquad r\vec{a} \notin E_\varphi\,, \quad r \geq r_0 > 0\,.$$

D'où le Théorème 5.

BIBLIOGRAPHIE

[1] G. Valiron, Sur les fonctions entières d'ordre nul et d'ordre fini et en particulier les fonctions à correspondance régulière. (Thèse, Paris 1912) *Ann. Toulouse.*

[2] W. K. Hayman, Slowly growing integral and subharmonic functions. *Comm. Math. Helvetica. Vol.* 34, 1960.

[3] P. Lelong, Sur l'étude des noyaux primaires et sur un théorème des fonctions entières de *n* variables. *C.R.* t. 237, pp. 1379.

[4] P. Lelong, Propriétés métriques des variétés analytiques complexes définies par une équation. *Ann. Sci. de l'Ec. Norm. Sup.* t. 67, 1950.

[5] P. Lelong, Les fonctions plurisousharmoniques. *Ann. Sci. de l'Ec. Norm. Sup.*, t. 62, 1945.

[6] V. Avanissian, Fonctions plurisousharmoniques et fonctions doublement sousharmoniques. (Thèse — Paris 1960). *Ann. Sci. de l'Ec. Norm. Sup.* t. 78, 1961.

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(Reçu le 10.IX.1961)

DETERMINANTS OF EIGENFUNCTIONS OF STURM-LIOUVILLE EQUATIONS

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§ 1. Introduction.

In a previous paper Szegö [1] and this author investigated oscillation properties of certain determinants whose elements were orthogonal polynomials. Two kinds of determinants were considered. The first concerned a generalization of the Turán type determinant; this satisfied a variety of inequalities applicable only in the case of the classical polynomials. The second set of determinantal inequalities refers to the Wronskian of the form

$$(1)\qquad W\begin{pmatrix} n_1, n_2, \dots, n_k \\ x_1, x_2, \dots, x_k \end{pmatrix} = \begin{vmatrix} Q_{n_1}(x_1) & Q_{n_2}(x_1) & \dots & Q_{n_k}(x_1) \\ Q_{n_1}(x_2) & Q_{n_2}(x_2) & \dots & Q_{n_k}(x_2) \\ \vdots & \vdots & & \vdots \\ Q_{n_1}(x_k) & Q_{n_2}(x_k) & \dots & Q_{n_k}(x_k) \end{vmatrix};$$

$$0 \leqq n_1 < n_2 < \dots < n_k, \quad x_1 \leqq x_2 \leqq \dots \leqq x_k,$$

whose elements are chosen from a family of orthogonal polynomials $\{Q_n(x)\}$ and x is a real variable.[1] The x values may be selected with equalities and in this case the determinant is to be interpreted by the convention that for repeated x values the respective columns are replaced by successive derivatives. For example, when

$$x_1 < x = x_2 = x_3 = \dots = x_r < x_{r+1} < \dots < x_k,$$

we have

1. Unless stated explicitly to the contrary the determinant (1) always has the rows and columns ordered so that x_i and n_j are arranged according to increasing values.

$$W\begin{pmatrix} n_1, n_2, \ldots, n_k \\ x_1, x_2, \ldots, x_k \end{pmatrix} = \begin{vmatrix} Q_{n_1}(x_1) & \ldots & Q_{n_k}(x_1) \\ Q_{n_1}(x) & \ldots & Q_{n_k}(x) \\ Q'_{n_1}(x) & \ldots & Q'_{n_k}(x) \\ \vdots & & \vdots \\ Q^{(r-2)}_{n_1}(x) & \ldots & Q^{(r-2)}_{n_k}(x) \\ Q_{n_1}(x_{r+1}) & \ldots & Q_{n_k}(x_{r+1}) \\ \vdots & & \vdots \\ Q_{n_1}(x_k) & \ldots & Q_{n_k}(x_k) \end{vmatrix}.$$

In this paper we study the Wronskian functions (1) where the polynomial system $Q_n(x)$ is replaced by a family of eigenfunctions

$$\varphi_0(x), \quad \varphi_1(x), \quad \varphi_2(x), \ldots$$

belonging to an appropriate linear operator. The main examples derive from an eigenvalue problem associated with a Sturm-Liouville differential operator. (We shall also discuss in Section 2 some cases of general linear transformations induced by kernel functions.) The Sturm-Liouville equation under consideration is stipulated to be of the form

$$\text{(2)} \qquad \frac{1}{\rho(x)}(\varkappa(x)\varphi'(x))' = -\lambda\varphi(x) \qquad a<x<b$$

plus suitable boundary conditions. The functions $\rho(x)$ and $\varkappa(x)$ are assumed to be positive and continuously differentiable on the interior of the interval (a, b). Also, we assume that at worst, b appears as a regular singular point while a is a bona-fide regular point. Further hypotheses and restrictions will be noted as needed.

In general, we shall consider Wronskians of the specific form:

$$\text{(3)} \quad W(\varphi_{n_1}(x), \varphi_{n_2}(x), \ldots, \varphi_{n_k}(x)) = \begin{vmatrix} \varphi_{n_1}(x) & \varphi_{n_2}(x) & \ldots \varphi_{n_k}(x) \\ \varphi'_{n_1}(x) & \varphi'_{n_2}(x) & \ldots \varphi'_{n_k}(x) \\ \vdots & \vdots & \vdots \\ \varphi^{(k-1)}_{n_1}(x) & \varphi^{(k-1)}_{n_2}(x) & \ldots \varphi^{(k-1)}_{n_k}(x) \end{vmatrix},$$

$$0 \leqq n_1 < n_2 < \ldots < n_k, \quad a < x < b.$$

This determinant is meaningful whenever each φ_i possesses $k-1$ derivatives. In the case where φ_i are solutions of (2) ($\varphi_i(x)$ are necessarily differentiable), then a modified Wronskian expression with wider applicability can be introduced. To this end, let λ_i denote the eigenvalue corresponding to φ_i, and suppose k is even. We construct the determinant

$$\tilde{W}(\varphi_{n_1}(x), \varphi_{n_2}(x), \ldots, \varphi_{n_k}(x)) \tag{4}$$

$$= \begin{vmatrix} \varphi_{n_1}(x) & \varphi_{n_2}(x) & \ldots & \varphi_{n_k}(x) \\ \varphi'_{n_1}(x) & \varphi'_{n_2}(x) & \ldots & \varphi'_{n_k}(x) \\ \lambda_{n_1}\varphi_{n_1}(x) & \lambda_{n_2}\varphi_{n_2}(x) & \ldots & \lambda_{n_k}\varphi_{n_k}(x) \\ \lambda_{n_1}\varphi'_{n_1}(x) & \lambda_{n_2}\varphi'_{n_2}(x) & \ldots & \lambda_{n_k}\varphi'_{n_k}(x) \\ \vdots & \vdots & & \vdots \\ \lambda_{n_1}^{(k-2)/2}\varphi_{n_1}(x) & \lambda_{n_2}^{(k-2)/2}\varphi_{n_2}(x) & \ldots & \lambda_{n_k}^{(k-2)/2}\varphi_{n_k}(x) \\ \lambda_{n_1}^{(k-2)/2}\varphi'_{n_2}(x) & \lambda_{n_2}^{(k-2)/2}\varphi'_{n_2}(x) & \ldots & \lambda_{n_k}^{(k-2)/2}\varphi'_{n_k}(x) \end{vmatrix}.$$

In other words, the row $(\varphi_{n_1}^{(l)}(x), \ldots, \varphi_{n_k}^{(l)}(x))$ is replaced by

$$\lambda_{n_1}^{(l-1)/2}\varphi_{n_1}(x),\ \lambda_{n_2}^{(l-1)/2}\varphi_{n_2}(x), \ldots, \lambda_{n_k}^{(l-1)/2}\varphi_{n_k}(x) \qquad \text{if } l \text{ is odd}$$

and by

$$\lambda_{n_1}^{(l-2)/2}\varphi'_{n_1}(x),\ \lambda_{n_2}^{(l-2)/2}\varphi'_{n_2}(x), \ldots, \lambda_{n_k}^{(l-2)/2}\varphi'_{n_k}(x) \qquad \text{if } l \text{ is even.}$$

If φ_i satisfies (2) corresponding to $\lambda = \lambda_i$ and is k times differentiable, and provided $\varkappa(x)$ and $\rho(x)$ are sufficiently differentiable, then a simple manipulation on the rows of (3) reveals that the two Wronskian functions $\tilde{W}$ and W are equal apart from a factor which is a power of $\varkappa/\rho$. However, (4) can be defined in many cases where (3) is not meaningful; for example where $\varkappa(x)$ is assumed only to be measurable. Henceforth, in dealing with the Wronskian functions W or $\tilde{W}$, we drop the $\sim$ symbol, the relevant interpretation will be clear from the context.

We will also have occasion to refer to a generalized Sturm-Liouville eigenvalue problem defined by the relation

$$D_\rho(D_\alpha \varphi) = -\lambda\varphi \tag{5}$$

coupled with appropriate boundary conditions. Here, D_α designates differentiation of φ with respect to the measure α and D_ρ has a similar connotation.

Since we deal with measures on the real line, it is convenient to think of α and ρ as increasing functions normalized to be continuous from the right and left respectively. Then

$$D_\alpha \varphi(x) = \lim_{h \to 0+} \frac{\varphi(x+h) - \varphi(x)}{\alpha(x+h) - \alpha(x)}$$

if α is continuous at x, otherwise

$$D_\alpha \varphi(x) = \frac{\varphi(x+) - \varphi(x-)}{\alpha(x+) - \alpha(x-)} .$$

When D_α is interpreted as differentiation on the right, D_ρ will refer to differentiation on the left.

Such differentiation operators were introduced and studied rather extensively by Feller [2] in connection with his investigations of generalized diffusion processes.

Orthogonal polynomials $\{Q_n(x)\}$ can be regarded as eigenfunctions associated with an equation of the form (5). We now indicate the necessary identifications validating this statement. Let us suppose that $\{Q_n(x)\}$ constitute a family of orthogonal polynomials on the interval $[0, a)$. Then $Q_n(x)$ obeys a recurrence law :

$$\begin{gathered} -xQ_n(x) = -(\nu_n+\mu_n)Q_n(x) + \nu_n Q_{n+1}(x) + \mu_n Q_{n-1}(x), \quad n \geqq 0 \\ Q_{-1}(x) \equiv 0, \quad Q_0(x) \equiv 1 \\ \nu_n > 0 \ (n \geqq 0), \quad \mu_n > 0 \ (n \geqq 1), \quad \mu_0 \geqq 0. \end{gathered} \tag{6}$$

Set $\pi_0=1$, $\pi_n=(\nu_0 \nu_1 \dots \nu_{n-1})/(\mu_1 \mu_2 \dots \mu_n)$, $(n \geqq 1)$ so that $\nu_n \pi_n = \pi_{n+1} \mu_{n+1}$, $n \geqq 0$, and define $\nu_{-1} \pi_{-1} = \mu_0$. Now, we rewrite (6) in the form

$$-xQ_n(x) = \frac{1}{\pi_n}\left(\frac{Q_{n+1}(x) - Q_n(x)}{\dfrac{1}{\nu_n \pi_n}} - \frac{Q_n(x) - Q_{n-1}(x)}{\dfrac{1}{\nu_{n-1}\pi_{n-1}}}\right), \quad n \geqq 0. \tag{7}$$

Introducing the notation $R_x(n) = Q_n(x)$, this relationship becomes

$$- xR_x(\cdot) = D_\pi(D_\alpha R_x(\cdot)) \tag{8}$$

where α represents the cumulative measure concentrated at the integers such that $d\alpha(n) = 1/\nu_{n-1}\,\pi_{n-1}$, $n = 0, 1, 2, \ldots$ and analogously $d\pi(n) = \pi_n$. It should be noted that in this case the argument of the eigenfuction is the index n of $Q_n(x)$ and the set of eigenvalues coincides with the spectrum (defined below) of the measure $\rho(x)$ with respect to which $Q_n(x)$ are orthogonal.[2] Let Λ denote the spectrum of ρ composed of all x values in every neighborhood of which ρ has positive measure. We introduce the concept of "successive x values" belonging to Λ. For this purpose we recall that a value x is an isolated point of Λ if there exists a neighborhood of x which intersects Λ only in x. A set of values $(x_1, x_2, \ldots, x_k)$ each contained in Λ is said to consist of "successive spectral values" provided either (a) all x_i are isolated points in Λ satisfying $x_1 < x_2 < \ldots < x_k$ and the union of the open intervals $\bigcup\limits_{i=1}^{k} (x_i, x_{i+1})$ has a void intersection with Λ, or (b) there exists an index r, $1 \leqq r < k$, such that $x_1 < x_2 < \ldots < x_{r-1}$ are successive isolated points in Λ in the sense of (a) and $x_r = x_{r+1} = \ldots = x_k$ is a non-isolated value of Λ immediately to the right of x_{r-1}, i.e.

$$(x_{r-1}, x_r) \cap \Lambda = \emptyset$$

We need one more definition. We say that the set S of $\{x_i\}$ satisfying $x_1 < x_2 < \ldots < x_k$ constitute a generalized even block (abbreviated G.E.B.) if k is even, and the set S splits into p $(p \geqq 1)$ groups

$$(x_1\, x_2 \ldots x_{\nu_1}),\ (x_{\mu_2} \ldots x_{\nu_2}),\ \ldots,\ (x_{\mu_p} \ldots x_{\nu_p})$$

where each individual group includes an even number of terms of "successive spectral values" and $x_{\nu_i} < x_{\mu_{i+1}}$ $(i = 1, \ldots, p-1)$.

We say that a set S of indices $(n_1, n_2, \ldots, n_k)$ constitute a G.E.B. if S divides into q groups

$$(n_1, n_1 + 1, \ldots, n_{\alpha_1})\,(n_{\beta_2}, n_{\beta_2} + 1, \ldots, n_{\alpha_2}) \ldots (n_{\beta_q}, n_{\beta_q} + 1, \ldots, n_{\alpha_q})$$

where each group consists of an even number of elements of consecutive integers and $n_{\alpha_\gamma} < n_{\beta_{\gamma+1}}$, $\gamma = 1, 2, \ldots, q-1$. A particular example is $q = 1$ and then $S = (n, n+1, \ldots, n+k-1)$ where k is even. The use of the

2. Assume that the orthogonal polynomials under consideration are associated with a unique measure.

common name (G.E.B.) for special blocks of indices and of "successive spectral values" should cause no difficulties. The following result of [1, Th. 3, p. 42] will serve crucially in the analysis of the generalized Wronskian functions (4).

Proposition A. For any system of orthogonal polynomials $\{Q_n(x)\}$, let $(x_1, x_2, \ldots, x_k)$ constitute a G.E.B. then

$$W\begin{pmatrix} n, n+1, \ldots, n+k-1 \\ x_1, x_2, \ldots, x_k \end{pmatrix} \neq 0, \quad n \geqq 0. \tag{9}$$

The assertion (9) is clearly independent of how the polynomial system has been normalized. We choose the normalization so that

$$Q_n(x) = a_n(-x)^n + \ldots, \qquad a_n > 0,$$

in which case then the sign in (9) is $(-1)^{k(k-1)/2}$ independent of n and of the choices of $\{x_i\}$ provided only they constitute a G.E.B.

In this paper we extend Proposition A to the case where eigenfunctions of a Sturm-Liouville problem are substituted in place of orthogonal polynomials. More precisely, let $\varphi_0, \varphi_1, \varphi_2, \ldots$ represent a complete family of eigenfunctions of equation (2); the coefficients $\varkappa(x)$ and $\rho(x)$ being sufficiently regular. (The exact conditions will be given in Section 3.) We will prove that if $n_1 < n_2 < \ldots < n_k$ constitute a G.E.B. then

$$W(\varphi_{n_1}(\xi), \varphi_{n_2}(\xi), \ldots, \varphi_{n_k}(\xi)) \neq 0 \quad \text{for} \quad a < \xi < b \tag{10}$$

where (10) is to be interpreted according to either (3) or (4) whichever is appropriate.

With the aid of (10) we further establish that the sequence of functions

$$\psi_n(x) = W(\varphi_n(x), \varphi_{n+1}(x), \ldots, \varphi_{n+k-1}(x)) \quad n = 0, 1, 2, \ldots \tag{11}$$

with k odd constitute a Sturm sequence. This means that $\psi_n(x)$ possesses precisely n simple zeros interior to (a, b) and the zeros of successive terms ψ_n and ψ_{n+1} strictly interlace.

Another sequence of functions which also exhibit the oscillation structure of a Sturm set is the augmented system:

$$\chi_n(x) = \begin{vmatrix} \varphi_l(x) & \varphi_l'(x) \\ \varphi_{n+l+1}(x), & \varphi'_{n+l+1}(x) \end{vmatrix}, \tag{12}$$

$$l \text{ fixed}, \quad n = 0, 1, 2, \ldots .$$

Their properties will be developed in Section 4. Other extensions of the results of [1] to the case of Sturm-Liouville systems will be indicated in Section 4.

Our principal method of proof is to discretize the Sturm-Liouville equation (2) so that it becomes the recurrence law of a finite system of orthogonal polynomials. We then invoke the result of Proposition A. Upon refining the approximating discrete system we obtain in the limit that (10) keeps a single sign. The proof that actually strict inequality prevails has to be dealt with by separate arguments. In Section 2 some weaker results in more general context are obtained by a more direct method.

§ 2. Totally Positive Integral Transformations and Associated Wronskian Determinants.

We consider the integral transformation

$$(Tf)(x) = \int_a^b K(x, y)\, f(y)\, d\mu(y) \tag{13}$$

where $K(x, y)$ defines a continuous kernel on $[a, b] \times [a, b]$ and $d\mu(y)$ is a measure which can be totally discrete, absolutely continuous or any combination of the two. In all cases we assume that $\mu(\cdot)$ possesses an infinite number of points of increase. Also, we require $-\infty < a < b < \infty$. The domain of T is taken to be $C[a, b]$ — the set of all functions continuous on $[a, b]$.

The kernel K of the transformation (13) is said to be (strict) totally positive TP (STP) if

$$K\begin{pmatrix} x_1, x_2, \dots, x_r \\ y_1, y_2, \dots, y_r \end{pmatrix} = \begin{vmatrix} K(x_1, y_1) & \dots & K(x_1, y_r) \\ K(x_2, y_1) & \dots & K(x_2, y_r) \\ \vdots & & \vdots \\ K(x_r, y_1) & \dots & K(x_r, y_r) \end{vmatrix} \geqq 0 \quad (>0) \tag{14}$$

for all $r \geqq 1$ and $a < \begin{matrix} x_1 < x_2 < \dots < x_r \\ y_1 < y_2 < \dots < y_r \end{matrix} < b$. A kernel K is said to be oscillating if K is TP and there exists an iterate

$$K^{(l)}(x, y) = \int K^{(l-1)}(x, z)\, K(z, y)\, d\mu(z)$$

which is STP.

The structure of oscillating kernels has been explored in considerable detail by Gantmacher and Krein [3] in connection with the study of eigenvalue problems of coupled mechanical systems and the sign variation characteristics of the associated eigenfunctions. They prove the following important result:

Proposition B. Let K represent an oscillating kernel with a countable discrete spectrum, then

(i) The transformation (13) possesses a countable set of positive eigenvalues

$$0 < \lambda_0 < \lambda_1 < \lambda_2 < \dots$$

and to each spectral value (eigenvalue) respectively corresponds an eigenfunction, unique apart from a multiplicative constant,

$$\varphi_0, \varphi_1, \varphi_2, \varphi_3, \dots$$

with the following properties;

(ii) If a_m, $m = l, l+1, \dots, k$, denote real constants not all zero, then

$$\sum_{m=l}^{k} a_m \varphi_m(x) = \varphi(x) \tag{15}$$

has at least l nodal zeros (these are zeros in every neighborhood of which the function changes sign) interior to $[a, b]$ and at most k distinct zeros in (a, b). Zeros of φ located at either a or b are not to be counted in this statement. If strict inequality in (14) holds throughout the closed interval, then the endpoints may be included in the statement (ii).

(iii) It follows from (ii) that $\varphi_n(x)$ has precisely n nodal zeros and no other zeros in (a, b). Moreover, (ii) implies that

$$\Delta\begin{pmatrix} \varphi_0, \varphi_1, \dots, \varphi_k \\ x_0, x_1, \dots, x_k \end{pmatrix} = \begin{vmatrix} \varphi_0(x_0) & \varphi_0(x_1) & \dots & \varphi_0(x_k) \\ \varphi_1(x_0) & \varphi_1(x_1) & \dots & \varphi_1(x_k) \\ \vdots & \vdots & & \vdots \\ \varphi_k(x_0) & \varphi_k(x_1) & \dots & \varphi_k(x_k) \end{vmatrix} \tag{16}$$

never vanishes for any selection $a < x_0 < x_1 < \dots < x_k < b$. For definiteness unless stated explicitly to the contrary we normalize φ_i (i.e. choose the free multiplicative constant) so that (16) has the sign $(-1)^{k(k-1)/2}$.

(iv) The zeros of two successive eigenfunctions φ_n and φ_{n+1} strictly interlace.

(The proof of this theorem is given in [3, Chap. 4].)

An important example of Proposition B is the kernel

$$K(x,y)=\begin{cases}\Theta(x)\ \chi(y) & a\leq x\leq y\leq b\\ \Theta(y)\ \chi(x) & a\leq y\leq x\leq b\end{cases}$$

where $\Theta(t)\,\chi(t)>0$ and $\Theta'(t)\,\chi(t)-\chi'(t)\,\Theta(t)>0$ for all $a<t<b$ and $\mu(t)=t$.

An important feature of totally positive kernels is their variation diminishing property:

Proposition C. If $K(x,y)$ is TP and $f(y)$ changes sign j times then $g(x)=\int K(x,y)\,f(y)\,d\mu(y)$ ($d\mu(y)$ is a non-negative sigma finite measure and the integral is assumed to converge absolutely) changes sign at most j times; moreover, if $g(x)$ actually changes sign j times, then $g(x)$ must exhibit the same arrangement of signs as the function $f(y)$ (3) when x and y traverse their respective domains from left to right.

By requiring a stronger version of the determinantal condition (14) we can bound the number of zeros of g, $Z(g)$, by the number of sign changes, $V(f)$, of the function f. In order to derive such bounds, we must introduce several additional concepts. As a preliminary, we note that if $K(x,y)$ belongs to C^{2m} (i.e., K is $2m$ fold continuously differentiable) it follows from (14) that

$$(17)\qquad \begin{vmatrix} K(x,y) & \dfrac{\partial K(x,y)}{\partial y} & \cdots & \dfrac{\partial^{m-1}K(x,y)}{\partial y^{m-1}} \\[2ex] \dfrac{\partial K(x,y)}{\partial x} & \dfrac{\partial^2 K(x,y)}{\partial x\,\partial y} & \cdots & \dfrac{\partial^m K(x,y)}{\partial x\,\partial y^{m-1}} \\ \vdots & & & \\ \dfrac{\partial^{m-1}K(x,y)}{\partial x^{m-1}} & \dfrac{\partial^m K(x,y)}{\partial^{m-1}x\,\partial y} & \cdots & \dfrac{\partial^{2m-2}K(x,y)}{\partial^{m-1}x\,\partial^{m-1}y}\end{vmatrix}\geq 0,$$

$$a<x,\ y<b.$$

3. The number of sign changes $V(f)$ of a real valued function f is $\sup\limits_{1\leq i\leq m} V(f(x_i))$ where $V(f(x_i))$ denotes the number of sign changes of the sequence $f(x_1), f(x_2), \dots, f(x_m)$ (zero values are discarded) with x_i selected arbitrarily from the domain of definition of f and arranged so that $x_1<x_2<\dots<x_m$; m is any positive integer.

The formal proof of (17) appears in [4, Section 1]; it involves simple use of the mean value theorem.

It is worth noting here that even if K is STP, the determinant (17) may vanish at exceptional points.

We now indicate an extension of (17). For this purpose, we must assign a meaning to the determinant

$$K\begin{pmatrix} x_1, x_2, \dots, x_m \\ y_1, y_2, \dots, y_m \end{pmatrix} \tag{18}$$

where $x_1 \leqq x_2 \leqq \dots \leqq x_m$ and $y_1 \leqq y_2 \leqq \dots \leqq y_m$ and distinguished in that several of the x's or y's can be equal. The convention we will employ is similar to that introduced in our discussion of Wronskians. When there is present in (18) a block of equal x values, the successive rows of (18), corresponding to these x values, are composed of the successive derivatives with respect to x; i.e., $K, \frac{\partial K}{\partial x}, \dots, \frac{\partial^{r-1} K}{\partial x^{r-1}}$ where r is the number of equal x values. Similar considerations apply in the case of blocks of equal y values with the appropriate y differentiations made on the successive columns.

Similar to (17), we obtain that if K is TP and possesses the necessary differentiability, then

$$K\begin{pmatrix} x_1, x_2, \dots, x_m \\ y_1, y_2, \dots, y_m \end{pmatrix} \geqq 0 \quad \text{for} \quad a \leqq \begin{matrix} x_1 \leqq x_2 \leqq \dots \leqq x_m \\ y_1 \leqq y_2 \leqq \dots \leqq y_m \end{matrix} \leqq b. \tag{18a}$$

This relation subsumes (17) which is the special case $x_1 = x_2 = \dots = x_m = x$ and $y_1 = y_2 = \dots = y_m = y$. We are now ready to propose a definition of total positivity in terms of derivatives of K. The function K is said to be extended totally positive in the x variable (abbreviated ETP(x)) if

$$K\begin{pmatrix} x, x, \dots, x \\ y_1, y_2, \dots, y_m \end{pmatrix} > 0 \quad \text{for all} \quad \begin{matrix} a < x < b \\ a < y_1 < y_2 < \dots < y_m < b. \end{matrix}$$

The emphasis is on strict inequality since the case of inequality including the possibility of equality constantly prevails if K is TP and C^{m-1} in x (see (18a)). Similarly we may define the concept of ETP(y). Finally we say that K is extended totally positive in both variables (written ETP with no reference to either the x or y variable) if

$$K\begin{pmatrix} x, & x, & \ldots, & x \\ y, & y, & \ldots, & y \end{pmatrix} > 0 \quad \text{for all} \quad a < x,\ y < b. \tag{19}$$

The definitions involving the concept of ETP are meaningful only if K is suitably continuously differentiable which we will always assume unless there is a statement to the contrary.

By suitably adapting the arguments of [4, p. 285] we may prove that if K is ETP(x) or ETP(y) then K is STP. Similarly, it follows that if K is ETP then

$$K\begin{pmatrix} x_1, & x_2, & \ldots, & x_m \\ y_1, & y_2, & \ldots, & y_m \end{pmatrix} > 0 \tag{20}$$

for all x_i, y_j satisfying the conditions of (18a). The detailed proof of this assertion, other extensions and the finer interplay of these ideas will be published elsewhere [7].

The above discussion sets forth four formulations of total positivity; namely TP, STP, ETP(x) or ETP(y) and ETP, each of which implies the preceding. All four versions are useful in different contexts.

For later purposes we now indicate briefly the variation diminishing properties of the transformation

$$g(x) = \int K(x, y)\, f(y)\, d\mu(y), \quad d\mu(y) \geqq 0\,. \tag{21}$$

where the function $K(x, y)$ is assumed to be totally positive in the various senses described above. For ease of exposition we postulate that f, g and K occurring in (21) are sufficiently continuous so that all operations performed on the integral are meaningful. The variation diminishing property (VDP) of (21) in the case of TP functions was already stated in Proposition C. If we now assume that K is ETP(x) and that sufficient differentiations under the integral sign are permissible, then the VDP statement can be strengthened as follows. Let $Z(g)$ denote the number of zeros, counting multiplicities, of the function g. If K is ETP(x), then

$$Z(g) \leqq V(f). \tag{22}$$

Finally, we shall need an intermediate concept petween STP and ETP. We say that $K(x, y)$ is quasi extended totally positive (abbreviated QTP) if

$$K\begin{pmatrix} x_1, x_2, \ldots, x_k \\ y_1, y_2, \ldots, y_k \end{pmatrix} > 0 \tag{23}$$

for any set of x's and y's satisfying $a < \begin{matrix} x_1 \leqq x_2 \leqq \ldots \leqq x_k \\ y_1 \leqq y_2 \leqq \ldots \leqq y_k \end{matrix} < b$, k arbitrary with no three x values or y values equal. The corresponding VDP is as follows: Let $\tilde{Z}(g)$ denote the number of zeros of g with zeros of multiplicity greater than 1 counted twice. Now if (21) is satisfied and K is QTP, then

$$\tilde{Z}(g) \leqq V(f). \tag{24}$$

We are now prepared to develop the principal theorem of this section.

Theorem 1. (a) Let $K(x, y)$ denote a continuous oscillating kernel on $[a, b] \times [a, b]$ such that for any integer k there exists an iterate $K^{(l)}(x, y)$ which is $2k$ times continuously differentiable and ETP of order k (i.e., satisfies (19) for all determinants of size $\leqq k$).

Let $\varphi_m(x)$ and λ_m, $m=0, 1, \ldots$, denote the eigenfunctions and eigenvalues guaranteed by Proposition B. (Since φ_m are eigenfunctions for any iterate of K, they are necessarily C^∞, i.e., infinitely differentiable.) Let k be even. Then the Wronskian function (3) never vanishes for $a < x < b$, i.e.,

$$W(\varphi_n(x), \varphi_{n+1}(x), \ldots, \varphi_{n+k-1}(x)) \neq 0, \quad a < x < b. \tag{25}$$

(b) Suppose $K(x, y)$ is absolutely continuous in each variable, has uniformly bounded difference quotients in each variable separately and the first partial derivative exist at all but a finite number of points[4]. Moreover, suppose that K satisfies the hypothesis of Proposition B and that some iterate of K enjoys properties (23) and (24), i.e., an iterate of K is QTP. Let k be even, then the generalized Wronskian function $\breve{W}(\varphi_n, \varphi_{n+1}, \ldots . \varphi_{n+k-1})$ introduced in (4) never vanishes for $a < x < b$.

4. The places where $\partial K/\partial x\ (x, y)$ fails to exist generally depend on y.

Proof : (a) Suppose to the contrary that (25) vanishes at x_0, $a < x_0 < b$. Then there exist real constants a_i, $0 \leqq i \leqq k-1$, not all zero, such that

$$a_0 \varphi_n(x) + a_1 \varphi_n(x) + \dots + a_{k-1} \varphi_{n+k-1}(x) = (x-x_0)^k L(x) . \tag{26}$$

Further $L(x)$ is not identically zero by property (ii) of Proposition B.

We observe that

$$g(x) = (x-x_0)^k L(x) = \int_a^b K^{(l)}(x, y) \Big[\sum_{\nu=0}^{k-1} a_\nu \frac{\varphi_{n+\nu}(y)}{\lambda_{n+\nu}^l} \Big] d\mu(y) \tag{27}$$

where $K^{(l)}$ is appropriately ETP of order $\leqq k$ and

$$f(x) = \sum_{\nu=0}^{k-1} a_\nu \frac{\varphi_{n+\nu}(x)}{\lambda_{n+\nu}^l}$$

is not identically zero. By virtue of property (ii) of Proposition B, we know that $V(f) \leqq n+k-1$. Comparing (27) with (21) and (22) we deduce that

$$Z(g) \leqq n+k-1,$$

and consequently $L(x)$ can vanish at most $n-1$ times. At this point we deduce immediately a contradiction of the fact that g must exhibit at least n nodal zeros, part (ii) Proposition B. We present a more elaborate proof which does not refer to (ii) whose method introduced by Krein and Gantmacher will be needed in part (b). Suppose, to obtain our contradiction that $L(x)$ actually changes sign $p \leqq n-1$ times. Let $x_1 < x_2 < \dots < x_p$ denote the points at which L changes sign. More precisely, say for definiteness that $L(x) \leqq 0$ for $x < x_1$ and somewhere in this interval <0, $L(x) \geqq 0$ on $x_1 < x < x_2$ and somewhere on this interval > 0, etc. We construct the function

$$u(x) = \begin{vmatrix} \psi_0(x_1) & \psi_1(x_1) & \dots & \psi_p(x_1) \\ \psi_0(x_2) & \psi_1(x_2) & \dots & \psi_p(x_2) \\ \vdots & \vdots & & \vdots \\ \psi_0(x_p) & \psi_1(x_p) & \dots & \psi_p(x_p) \\ \psi_0(x) & \psi_1(x) & \dots & \psi_p(x) \end{vmatrix} \tag{28}$$

where $\psi_i(y)$ represent the eigenfunctions of the adjoint transformation :

$$\lambda_j \psi_j(y) = \int_a^b K(x, y)\, \psi_j(x)\, d\mu(x)\,. \tag{29}$$

Clearly, the transpose kernel defining (29) possesses the same properties as K and of course the oscillation characteristics of the system $\{\psi_j(y)\}$ are identical to those of $\{\varphi_i(x)\}$. Moreover, the eigenvalues of the integral equation (29) and that of (13) coincide. It is a standard result that

$$\int_a^b \psi_j(\xi)\, \varphi_i(\xi)\, d\mu(\xi) = 0 \quad \text{for } i \neq j\,. \tag{30}$$

In view of property (ii) of Proposition B (see also (16)), we infer that $u(x)$ strictly alternates in sign as we traverse the successive intervals

$$[x_i, x_{i+1}], \quad i = 0, 1, \ldots, p;$$

for completeness we define $x_0 = a$ and $x_{p+1} = b$. It follows that $u(x) \cdot L(x)$ keeps a constant sign throughout (a, b) and vanishes at most at $n-1$ points. Since k is even, we conclude that

$$\int_a^b u(x)\, (x - x_0)^k L(x)\, d\mu(x) \neq 0\,. \tag{31}$$

On the other hand, we note that $u(x)$ is a linear combination of $\psi_0, \psi_1, \ldots, \psi_{n-1}$ while $(x-x_0)^k L(x)$ is a linear combination of $\varphi_n, \varphi_{n+1}, \ldots, \varphi_{n+k-1}$. By the orthogonality property (30), we obtain

$$\int_a^b u(x)\, (x - x_0)^k L(x)\, d\mu(x) = 0$$

in contradiction to (31). This completes the proof of statement (a).

As a preliminary to the proof of part (b), we need the following lemma:

Lemma. *Let the conditions of Theorem 1, part (b), prevail. Then each zero x_0 $(a < x_0 < b)$ of $b\varphi_m + c\varphi_{m+1}$ $(b^2 + c^2 > 0)$ is of multiplicity 1.*

Proof: The proof is by induction on m. Consider the case $m = 0$. Choose $l \geqq 0$ so that

$$K^{(l)}\begin{pmatrix} x & , x \\ y_1 & , y_2 \end{pmatrix} > 0 \quad \text{for all} \quad a < y_1 < y_2 < b \text{ and } a < x < b.$$

A trivial calculation gives

$$\lambda_0^l \lambda_1^l \begin{vmatrix} \varphi_0(x) & \varphi_1(x) \\ \varphi'_0(x) & \varphi'_1(x) \end{vmatrix} = \iint\limits_{y_1 \ \ y_2} K^{(l)}\begin{pmatrix} x & x \\ y_1 & y_2 \end{pmatrix} \Delta \begin{pmatrix} \varphi_0 & \varphi_1 \\ y_1 & y_2 \end{pmatrix} d\mu(y_1)\, d\mu(y_2).$$

(For the meaning of the symbol Δ, see equation (16).) By virtue of property (ii) of Proposition B, and taking account of the specific way in which the φ_i have been normalized, we obtain the inequality $W(\varphi_0, \varphi_1) < 0$. This is the desired result for $m = 0$. Suppose now we have proved our assertion for all $m < r$. We proceed to advance the induction. Assume to the contrary that

$$h(x) = c\varphi_r(x) + d\varphi_{r+1}(x) = (x - x_0)^2 M(x), \quad a < x_0 < b,$$

and c and d are neither zero. By property (ii) of Proposition B, $M(x)$ has at least r nodal zeros. Now in the case that $M(x)$ does not vanish at x_0 or that x_0 is a non-nodal zero, we add the function $\varepsilon\varphi_r(x)$ with $|\varepsilon|$ sufficiently small to $h(x)$, thus producing $h^*(x) = c'\varphi_r(x) + d\varphi_{r+1}(x)$ exhibiting at least $r + 2$ nodal zeros; this contradicts (ii). In the other contingency that $M(x)$ has a nodal zero at x_0, we construct $\varepsilon(\gamma\varphi_{r-1}(x) + \delta\varphi_r(x))$, vanishing at x_0 exactly of first order, which is possible by the induction hypothesis. The magnitude $|\varepsilon|$ is kept small so that

$$\varepsilon\gamma\varphi_{r-1}(x) + \varepsilon\delta\varphi_r(x) + c\varphi_r(x) + d\varphi_{r+1}(x)$$

now has three nodal zeros in the neighborhood of x_0, at the same time preserving, apart from small displacement, the other zeros. Again we obtain a contradiction of property (ii) and the lemma is proved.

Proof of (b). Suppose the statement of (b) is false, i.e., (4) vanishes at x_0 in (a, b). Then there exist real constants a_ν not all zero such that

$$(32) \qquad a_0\varphi_n(x) + a_1\varphi_{n+1}(x) + \dots + a_{k-1}\varphi_{n+k-1}(x) = (x - x_0)^2 L_0(x)$$
$$a_0\lambda_n\varphi_n(x) + a_1\lambda_{n+1}\varphi_{n+1}(x) + \dots + a_{k-1}\lambda_{n+k-1}\varphi_{n+k-1}(x) = (x - x_0)^2 L_1(x)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_0\lambda_n^{(k-2)/2}\varphi_n(x) + a_1\lambda_{n+1}^{(k-2)/2}\varphi_{n+1}(x) + \dots + a_{k-1}\lambda_{n+k-1}^{(k-2)/2}\varphi_{n+k-1}(x) = (x - x_0)^2 L_{(k-2)/2}(x).$$

We claim that $V(L_0) \leqq n + k - 3$. This will follow if we establish that a zero of multiplicity > 1 for L_0 at x_0 is counted as two zeros when it is

a non-nodal zero and three in case that it is a nodal zero. The first contingency is proved by methods analogous to those used in dealing with equation (27). We omit the details.

The analysis of the second contingency depends on the arguments of the lemma. We follow the method of the lemma and add a suitable linear combination of φ_n and φ_{n+1} (small in magnitude) to the first equation and produce three nodal zeros in the neighborhood of x_0, while essentially keeping the other nodal zeros intact aside from small displacements. This fact combined with the assertion of property (ii) of Proposition B easily implies that $V(L_0) \leqq n + k - 3$.

Since

$$g_1(x) = (x - x_0)^2 L_1(x) = \int_a^b K(x, y)(y - x_0)^2 L_0(y)\, d\mu(y),$$

we conclude that $\tilde{Z}(g_1) \leqq n + k - 3$. The above argument applied again shows now that $V(L_1) \leqq n + k - 5$. We repeat this method on the successive functions of (32) and this culminates in the result

$$V(L_{(k-2)/2}) \leqq n - 1 .$$

However, this inequality leads to a contradiction in the same manner as in part (a) of the theorem. The proof of the theorem is now complete.

Examination of the proof and statement of Theorem 1 suggests various extensions which we will not pursue. For example, we could formulate an intermediate type Wronskian involving derivatives up to a certain order analogous to the formation of (4) out of (3). We shall not enter into the statement of Theorem 1, corresponding to these possibilities.

Some corollaries of Theorem 1 concerned with constructing various Sturm sequences based on the system $\{\varphi_n(x)\}$ will be described in Section 4.

We do not know whether the conclusion (25) can be extended to the case where the index set $n_1 < n_2 < \ldots < n_k$ is a GEB rather than the special case of a block of consecutive indices. For the special class of TP kernels of the form

$$K(x, y) = \begin{cases} \varphi(x)\ \psi(y) & a \leqq x \leqq y \leqq b \\ \varphi(y)\ \psi(x) & a \leqq y \leqq x \leqq b, \end{cases}$$

the stronger assertion in terms of a GEB index set is indeed valid (see Remark 5 of the following section).

§ 3. Generalized Wronskians of Eigenfunctions of Sturm-Liouville Operators.

In this section we examine the Wronskian function (4) whose elements are eigenfunctions belonging to the Sturm-Liouville operator

$$\frac{1}{\rho(x)}(\varkappa(x)\varphi'(x))' = -\lambda\varphi(x) \quad a<x<b \tag{33}$$

coupled with an appropriate boundary condition. We require that $\varkappa(x) > 0$, and $\rho(x) > 0$ throughout $a < x < b$. Suppose the boundary value problem admits only a countable spectrum of simple eigenvalues $\lambda_0 < \lambda_1 < \lambda_2 < \dots$ and corresponding eigenfunctions $\varphi_0, \varphi_1, \varphi_2, \dots$ which have been normalized in some definite way. Although the results of Theorem 2 below are valid in considerable generality, we give detailed consideration only on the case where the endpoints a and b are both regular points of (33). (Some discussion of the general Sturm-Liouville operator will be indicated at the close of this section). Specifically, we require that $\varkappa(x) > 0$ and $\rho(x) > 0$ on the closed interval $[a, b]$. The initial condition is postulated to be

$$\varphi(a) = 1 \quad \text{and} \quad \varphi'(a) = 0\,. \tag{34}$$

The boundary condition is assumed to be of the form

$$\alpha\varphi(b) + \beta\varkappa(b)\varphi'(b) = 0 \tag{35}$$

where α, β denote fixed real constants.

We construct a discrete approximation to the system (33) as follows: For each fixed N, the interval $[a, b]$ is divided into $N+1$ equal parts;

$$x_i^N = a + \frac{i}{N+1}(b-a)\,, \quad 0 \leq i \leq N+1\,.$$

Now we determine γ_m^N and μ_m^N so that

$$\frac{1}{\gamma_m^N \pi_m^N} = \int_{x_m^N}^{x_{m+1}^N} \frac{dx}{\varkappa(x)} \qquad 0 \leq m \leq N, \tag{36}$$

and

$$\pi_0^N = 1 , \ \pi_m^N = \int_{x_{m-1}^N}^{x_m^N} \rho(x)\, dx \qquad 0 < m \leq N ,$$

where

$$\pi_m^N = \frac{\nu_0^N \nu_1^N \ \dots \ \nu_{m-1}^N}{\mu_1^N \mu_2^N \ \dots \ \mu_m^N} , \qquad m = 1, 2, \dots , N .$$

Consider the finite recurrence law

$$-\nu\, Q_m^N(\lambda) = -(\nu_m^N + \mu_m^N)\, Q_m^N(\lambda) + \nu_m^N\, Q_{m+1}^N(\lambda) + \mu_m^N\, Q_{m-1}^N(\lambda) \qquad m = 0, 1, \dots , N \tag{37}$$

where $Q_{-1}^N(\lambda) \equiv 0$, $Q_0^N(\lambda) \equiv 1$ and $\mu_0^N = 0$.

We now describe the nature of the boundary condition for this discrete system. As is traditional, a derivative operation is replaced by a difference operation. Thus, a discretized version of (35) reduces to a condition of the form

$$\alpha^N Q_N^N(\lambda) + \beta^N Q_{N+1}^N(\lambda) = 0 . \tag{38}$$

The polynomial on the left-hand side of (38) represents a quasi orthogonal polynomial [5, Chap. 3] of degree $N+1$ and therefore possesses $N+1$ simple real roots

$$\lambda_0^N < \lambda_1^N < \dots < \lambda_N^N . \tag{39}$$

Also, there exists a related finite measure σ^N with mass points located exclusively at (39) which comprises the spectrum of the finite system of polynomials $\{Q_m^N(\lambda)\}$. The polynomial system $\{Q_m^N\}_{m=0}^N$ is orthogonal with respect to σ^N .

We write (37) in the form (8). Now, by a standard inversion process we may convert this relation into a discrete Volterra integral equation;

$$Q_{m+1}^N(\lambda) = 1 - \lambda \sum_{k=0}^{m} \frac{1}{\nu_k^N \pi_k^N} \sum_{l=0}^{k} \pi_l^N Q_l^N(\lambda) , \qquad 0 \leq m \leq N \tag{40}$$

and

$$Q_0^N(\lambda) \equiv 1 .$$

This relation should be compared with the Volterra integral equation

$$\text{(41)} \qquad \varphi(x, \lambda) = 1 - \lambda \int_a^x \frac{d\xi}{\varkappa(\xi)} \int_a^\xi \rho(\eta)\, \varphi(\eta, \lambda)\, d\eta$$

equivalent to (33) in that the solution of the latter subject to the initial conditions $\varphi(a, \lambda) = 1$, and $\varphi'(a, \lambda) = 0$ satisfies (41) and conversely. The solution of the boundary value problem (40) and the integral equation (41) are manifestly both unique.

It is relatively easy to show with the aid of (40) that $Q_{m+1}^N(\lambda)$ constitute an equi-continuous family (the index variable is N) of bounded functions converging uniformly to $\varphi(x, \lambda)$ as N increases to ∞ where m is adjusted to N so that $x \sim a + \frac{m}{N}(b-a)$.

It is also an easy consequence of (40) that

$$\text{(42)} \qquad N[Q_{m+1}^N(\lambda_i^N) - Q_m^N(\lambda_i^N)] \to \varphi'_i(x; \lambda)$$

provided m and N simultaneously increase to infinity preserving the asymptotic condition $x \sim a + \frac{(b-a)}{N+1} m$ as required above. The recurrence law (37) can be written in the form $D_{\rho^N} D_{\alpha^N} Q = -\lambda Q$ where α^N and ρ^N represent discrete measures determined as follows:

$$d\alpha_j^N = \frac{1}{\gamma_j^N \pi_j^N}, \quad d\rho_j^N = \pi_j^N .$$

Letting $N \to \infty$ and $m \to \infty$ as above we obtain as the limiting form of (37)

$$\text{(43)} \qquad \frac{1}{\rho(x)} (\varkappa(x)\, \varphi'(x, \lambda))' = -\lambda\, \varphi(x, \lambda) .$$

Naturally, the boundary condition at b is only satisfied for $\lambda = \lambda_i$ and then $\varphi(x, \lambda_i) = \varphi_i(x)$.

We are now ready to exploit Proposition A in the case of general

orthogonal polynomials in order to deduce analogous determinantal inequalities involving the eigenfunctions $\varphi_i(x)$.

Let

$$(\lambda_{i_1}^N, \lambda_{i_2}^N, \ldots, \lambda_{i_k}^N)$$

denote a GEB of spectral points. According to Proposition A we have that

$$(44) \qquad (-1)^{k(k-1)/2} W\begin{pmatrix} m, m+1, \ldots, m+k-1 \\ \lambda_{i_1}^N, \lambda_{i_2}^N, \ldots, \lambda_{i_k}^N \end{pmatrix}$$

$$= (-1)^{k(k-1)/2} \begin{vmatrix} Q_m^N(\lambda_{i_1}^N) & Q_m^N(\lambda_{i_2}^N) & \cdots & Q_m^N(\lambda_{i_k}^N) \\ Q_{m+1}^N(\lambda_{i_1}^N) & Q_{m+1}^N(\lambda_{i_2}^N) & \cdots & Q_{m+1}^N(\lambda_{i_k}^N) \\ \vdots & & & \\ Q_{m+k-1}^N(\lambda_{i_1}^N) & Q_{m+k-1}^N(\lambda_{i_2}^N) & \cdots & Q_{m+k-1}^N(\lambda_{i_k}^N) \end{vmatrix} > 0 .$$

The sign of (44) is explicit due to the normalization condition $Q_n^N(0) = 1$ which is satisfied by the nature of the recurrence law (37).

By performing obvious algebraic operations on the rows of (44) and repeatedly invoking the recursion law (37), we arrive at a determinant whose first column, apart from certain positive factors common to various rows, is

$$(45) \qquad \{Q_m(\lambda_{i_1}), N[Q_{m+1}(\lambda_{i_1}) - Q_m(\lambda_{i_1})], \lambda_{i_1} Q_{m+1}(\lambda_{i_1}),$$
$$\lambda_{i_1} N[Q_{m+2}(\lambda_{i_1}) - Q_{m+1}(\lambda_{i_1})], \ldots, \lambda_{i_1}^{(k-2)/2} Q_{m+(k-2)/2}(\lambda_{i_1}),$$
$$\lambda_{i_1}^{(k-2)/2} N[Q_{m+(k/2)}(\lambda_{i_1}) - Q_{m+(k/2)-1}(\lambda_{i_1})]\} .$$

The other columns are obtained from the first by replacing i_1 by $i_2, i_3, \ldots, i_k$, respectively. The superscript N was suppressed in (45) for typographical reasons to avoid a proliferation of superscripts.

Let m be selected so that

$$(46) \qquad a + \frac{(b-a)m}{N+1} \sim x$$

where x is fixed but arbitrary in $[a, b]$.

Now let $N \to \infty$ and $m \to \infty$ satisfying (46). By virtue of the convergence properties $Q_m^N(\lambda_i^N) \to \varphi_i(x)$ and (42), we obtain from (44)

$$(-1)^{k(k-1)/2}\, W \begin{pmatrix} \varphi_{i_1}, \varphi_{i_2}, \dots, \varphi_{i_k} \\ x \ , x \ , \dots , x \end{pmatrix} \geq 0 \tag{47}$$

where $(i_1, i_2, \dots, i_k)$ is a GEB. Here (47) is interpreted in the generalized sense of (4).

We now state the main theorem of this section.

Theorem 2. Let $(i_1, i_2, \dots, i_k)$ be a GEB and let $\{\varphi_i(x)\}$ denote a complete family of eigenfunctions satisfying (33) under the conditions (34) and (35), then

$$R(x) = (-1)^{k(k-1)/2}\, W \begin{pmatrix} \varphi_{i_1}, \varphi_{i_2}, \dots, \varphi_{i_k} \\ x \ , x \ , \dots , x \end{pmatrix} > 0 \tag{48}$$

for all $a<x<b$.

Proof: We have already proved (48) in the weak form (47) with equalities allowed. It remains to establish the strict inequality.

The proof will proceed by induction on the size of the determinant. Suppose we have proved (48) for all Wronskian determinants of the same kind of order at most $k-2$ (recall that k is even). We can handle $k=2$ by the same methods.

Suppose to the contrary that (48) vanishes at some point x_0 interior to the interval $[a, b]$. As a consequence, the row vectors are linearly dependent. We may express this dependence as follows. There exists a set of real constants not all zero which we regard as coefficients of the polynomials P_1 and P_2 such that

$$P_1(\lambda_{i_\nu})\, \varphi_{i_\nu}(x_0) + P_2(\lambda_{i_\nu})\, \varphi'_{i_\nu}(x_0) = 0\,, \qquad \nu = 1, 2, \dots, k \tag{49}$$

where $P_1(\lambda)$ and $P_2(\lambda)$ are polynomials of degree not exceeding $(k-2)/2$, at least one of which is nontrivial. Moreover, at least one of these polynomials must actually be of precise degree $(k-2)/2$, since otherwise the induction hypothesis is contradicted.

In view of (47), we infer that x_0 is a non-nodal zero of $[\varkappa(x)]^{k/2} R(x)$. We will see below that $[\varkappa(x)]^{k/2} R(x)$ is differentiable everywhere in (a, b). Then

$$\frac{1}{[\rho(x)]^{k/2}} \; \frac{d}{dx} \{[\varkappa(x)]^{k/2} R(x)\} \tag{50}$$

vanishes at x_0. (In calculating (50) we introduce the factor $\varkappa(x)$ in each row of derivatives, and then execute the indicated differentiation.) Appealing to the differential equation (33), we find that (50) is identical with the determinant $R(x)$ except in that the last row is replaced by the elements

$$\lambda_{i_1}^{k/2}\varphi_{i_1}(x_0)\ ,\ \lambda_{i_2}^{k/2}\varphi_{i_2}(x_0)\,,\ \ldots,\ \lambda_{i_k}^{k/2}\varphi_{i_k}(x_0)\,.$$

The vanishing of (50) implies the existence of real constants not all zero and associated polynomials with the property that

$$P_3(\lambda_{i_\nu})\,\varphi_{i_\nu}(x_0)+P_4(\lambda_{i_\nu})\,\varphi'_{i_\nu}(x_0)=0 \qquad \nu=1,2,\ldots,k\,. \tag{51}$$

The deduction of (51) parallels that of (49).

Here, $P_4(\lambda)$ is a polynomial of degree at most $\dfrac{k}{2}-2$ and $P_3(\lambda)$ denotes a polynomial of degree at least $\dfrac{k}{2}-1$ and not exceeding $k/2$. This follows from the induction hypothesis.

We now consider three cases:

(a) $\deg P_2=\dfrac{k}{2}-1$

(b) $\deg P_3=\dfrac{k}{2}$, $\deg P_2<\dfrac{k}{2}-1$

(c) $\deg P_3=\dfrac{k}{2}-1$ and $\deg P_2<\dfrac{k}{2}-1$.

Case (a): Since $\varphi_{i_\nu}(x_0)$ and $\varphi'_{i_\nu}(x_0)$ cannot simultaneously vanish, it follows by eliminating these variables from (49) and (51) that

$$P_3(\lambda)\,P_2(\lambda)-P_1(\lambda)\,P_4(\lambda)=0 \quad \text{for} \quad \lambda=\lambda_{i1}\,,\lambda_{i_2}\,,\ldots,\lambda_{i_k}\,. \tag{52}$$

The polynomial of the left-hand side of (52) is certainly nontrivial since $\deg[P_3\cdot P_2]\geqq k-2$, while $\deg P_1P_4\leqq k-3$. However (52) is of degree not exceeding $\dfrac{k}{2}+\dfrac{k}{2}-1=k-1$ and there exist k distinct zeros, an absurdity. This contradiction proves the theorem in this case.

Case (b): We derive the conclusion (52) as previously. It remains to verify that the left-hand polynomial of (52) is nontrivial. Consider the contrary event that (52) is identically zero. In particular,

$$\deg P_3 P_2 = \deg P_1 P_4 . \tag{53}$$

This relation formally states that

$$\frac{k}{2} l = \left(\frac{k}{2} - 1\right) m , \tag{54}$$

where $m = \deg P_4$ and $\deg P_2 = l$; $\deg P_1 = \frac{k}{2} - 1$ by the induction hypothesis since $\deg P_2 < \frac{k}{2} - 1$ by assumption. We may write this in the form $m = \frac{k}{2}(m - l)$. Since $m - l$ is a non-negative integer and because $m \leqq \frac{k}{2} - 2$, we have a contradiction unless $m = l = 0$. Thus, our discussion of case (b) is complete except for the possibility $m = 0$. We now show that this contingency also leads to a contradiction. If $l = 0$, then (49) reduces to

$$P_1(\lambda_{i_\nu}) \varphi_{i_\nu}(x_0) = 0 , \qquad \nu = 1, 2, \ldots, k . \tag{55}$$

By virtue of the hypothesis to the effect that $(i_1, i_2, \ldots, i_k)$ constitutes a GEB combined with the fact that two consecutive eigenfunctions cannot share common zeros, it follows that among the set of k distinct eigenvalues $\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_k}$ at least $k/2$ are zeros of $P_1(\lambda)$. But, $P_1(\lambda)$ is non-trivial of degree $\frac{k}{2} - 1$ which is clearly incompatible with the presumed count of the zeros.

Case (c): The stipulations of this case compel the further relations

$$\deg P_1 = \frac{k}{2} - 1 , \quad \deg P_4 \leqq \frac{k}{2} - 2 . \tag{56}$$

By assumption $\deg P_3 = \frac{k}{2} - 1$. The only situation which doesn't lead to an immediate contradiction by paraphrasing the analysis of that of case (a) is the possibility

$$\deg P_2 = \deg P_4 \leqq \frac{k}{2} - 2 . \tag{57}$$

It remains to examine this case. To this end, we prove first that under the conditions (56), (49) and (51), x_0 is a root of multiplicity at least three. The precise sense of the multiplicity is given below. We have already determined the form of the determinant

$$\frac{(\varkappa^{k/2}(x)\, R(x))'}{[\rho(x)]^{k/2}} = S(x)\,.$$

(See the comment immediately following (50).)

Now, consider

$$\frac{1}{[\rho(x)]^{(k/2)-1}} \quad \frac{d}{dx}\{\varkappa(x)^{(k/2)-1}\, S(x)\} = T(x)$$

calculated in a manner analogous to the calculation of (50). In this case, the determinant $T(x)$ reduces to a sum of two determinants $T_1(x)$ and $T_2(x)$. We describe the first column of each, the other columns are exhibited by replacing $i_2, i_3, \ldots, i_k$ for i_1 respectively.

$$T_1(x)$$
$$\sim [\varphi_{i_1}(x), \varphi'_{i_1}(x), \ldots, \lambda_{i_1}^{(k/2)-2}\varphi_{i_1}(x), \lambda_{i_1}^{(k/2)-2}\varphi'_{i_1}(x), \lambda_{i_1}^{(k/2)-1}\varphi'_{i_1}(x), \lambda_{i_1}^{(k/2)}\varphi_{i_1}(x)]$$

$$T_2(x)$$
$$\sim [\varphi_{i_1}(x), \varphi'_{i_1}(x), \ldots, \lambda_{i_1}^{(k/2)-2}\varphi_{i_1}(x), \lambda_{i_1}^{(k/2)-2}\varphi'_{i_1}(x), \lambda_{i_1}^{(k/2)-1}\varphi_{i_1}(x), \lambda_{i_1}^{k/2}\varphi'_{i_1}(x)]\,.$$

Since $\deg P_4 \leqq \frac{k}{2} - 2$ and $\deg P_3 = \frac{k}{2} - 1$ we infer on the basis of the assumption that (48) is zero at x_0 the existence of non-trivial constants exhibiting the rows of $T_2(x_0)$ as linearly dependent. Consequently, $T_2(x_0) = 0$. Utilizing (49) and since $\deg P_2 \leqq \frac{k}{2} - 2$ we can convert $T_1(x_0)$ into a determinant $T_1^{[1]}(x_0)$ such that the row

$$(\lambda_{i_1}^{(k/2)-2}\varphi_{i_1}(x_0)\,,\ \lambda_{i_2}^{(k/2)-2}\varphi_{i_2}(x_0)\,, \ldots)$$

is replaced by

$$(\lambda_{i_1}^{(k/2)-1}\varphi_{i_1}(x_0)\,,\ \lambda_{i_2}^{(k/2)-1}\varphi_{i_2}(x_0)\,, \ldots)$$

provided $a_{(k/2)-2} \neq 0$, where

$$P_1(\lambda) = a_0 + a_1\lambda + \ldots + a_{(k/2)-2}\lambda^{(k/2)-2} + a_{(k/2)-1}\lambda^{(k/2)-1}\,, \qquad a_{(k/2)-1} \neq 0\,.$$

If $a_{(k/2)-2} = 0$ we leave $T_1(x_0)$ unaltered. In either event $T_1(x_0)$ and the transformed determinant $T_1^{[1]}(x_0)$ differ by a fixed non-zero multiplicative factor. If $a_{(k/2)-2} \neq 0$ and $a_{(k/2)-3} \neq 0$, we transform $T_1^{[1]}(x_0)$ in the same manner as above to a determinant $T_1^{[2]}(x_0)$ where the row

$$(\lambda_{i_1}^{(k/2)-3} \varphi_{i_1}(x_0), \lambda_{i_2}^{(k/2)-3} \varphi_{i_2}(x_0), \ldots)$$

is replaced by

$$(\lambda_{i_1}^{(k/2)-2} \varphi_{i_1}(x_0), \lambda_{i_2}^{(k/2)-2} \varphi_{i_2}(x_0), \ldots).$$

If $a_{(k/2)-3} = 0$ we leave $T_1^{[1]}(x_0)$ unchanged. We continue in this way until we either reach a zero coefficient $a_l = 0$ (i.e., l is the largest index of a zero coefficient) or all the rows of the form

$$(\lambda_{i_1}^{l} \varphi_{i_1}(x_0), \lambda_{i_2}^{l} \varphi_{i_2}(x_0), \ldots), \qquad 0 \leqq l \leqq \frac{k}{2} - 2$$

have had exponents of λ_{i_ν} raised by one, resulting respectively in the form

$$(\lambda_{i_1}^{l+1} \varphi_{i_1}(x_0), \lambda_{i_2}^{l+1} \varphi_{i_2}(x_0), \ldots) \qquad 0 \leqq l \leqq \frac{k}{2} - 2.$$

Let us denote the final determinant obtained by this process as $\tilde{T}_1(x_0)$.

The relationship (49) or equivalently

$$\lambda_{i_\nu} P_1(\lambda_{i_\nu}) \varphi_{i_\nu}(x_0) + \lambda_{i_\nu} P_2(\lambda_{i_\nu}) \varphi'_{i_\nu}(x_0) = 0, \qquad \nu = 1, \ldots, k,$$

plainly applies also to the rows of $\tilde{T}_1(x_0)$; hence $\tilde{T}_1(x_0) = 0$. This completes the proof of the statement that x_0 is a root of multiplicity $\geqq 3$. But x_0 is a root of even multiplicity since (47) holds and so

$$\frac{1}{[\rho(x)]^{k/2}} \frac{d}{dx} [\varkappa^{k/2}(x) T(x)] = U(x) \tag{58}$$

also vanishes at x_0. By repeating the kind of analysis we have just finished, we find that $\dfrac{U(x_0)}{2}$ reduces to a single determinant whose first column has the form

$$[\varphi_{i_1}(x_0), \varphi'_{i_1}(x_0), \ldots, \lambda_{i_1}^{(k/2)-2} \varphi_{i_1}(x_0), \lambda_{i_1}^{(k/2)-2} \varphi'_{i_1}(x_0),$$
$$\lambda_{i_1}^{(k/2)-1} \varphi'_{i_1}(x_0), \lambda_{i_1}^{k/2} \varphi'_{i_1}(x_0)].$$

Since $U(x_0) = 0$ we obtain the identities

$$P_5(\lambda_{i_\nu})\varphi_{i_\nu}(x_0) + P_6(\lambda_{i_\nu})\varphi'_{i_\nu}(x_0) = 0\,, \qquad \nu = 1, \ldots, k\,, \tag{59}$$

where $\deg P_5 \leqq \frac{k}{2} - 2$ and $\frac{k}{2} - 1 \leqq \deg P_6 \leqq \frac{k}{2}$, the last being true because of the induction hypothesis.

We combine (51) and (59) (in the analogous manner to that of (52)) and arrive at a contradiction as in case (a). This finishes the discussion of case (c).

To complete the proof of the theorem we need to examine the case $k = 2$. We observe that

$$W\begin{pmatrix} \varphi_n\,, & \varphi_{n+1} \\ x\,, & x \end{pmatrix} = \begin{vmatrix} \varphi_n & \varphi_{n+1} \\ \varphi'_n & \varphi'_{n+1} \end{vmatrix} \qquad a < x < b$$

cannot be identically zero since in the contrary event φ_n is a multiple of φ_{n+1} which is impossible. From here on the arguments proceed as with the general size determinant; the steps are much simpler and will therefore be omitted. The proof of the theorem is now complete.

Remark 1: In forming the discrete approximation to (33), we considered only the boundary conditions (35) and the special initial condition (34). It is important to emphasize that the result stated in Theorem 2 is valid for the general initial condition

$$\gamma\varphi(a) + \varphi'(a) = 0\,, \quad \varphi(a) = 1\,, \quad \gamma \text{ real.} \tag{60}$$

Actually, this formulation can be reduced to the condition of (34) by considering the function $e^{\gamma x}\varphi(x) = \psi(x)$ where φ satisfies (33). The Sturm-Liouville equation for the function ψ is of the form

$$(\tilde{\varkappa}(x)\,\psi'(x))' + q(x)\,\psi(x) = -\lambda\tilde{\rho}(x)\,\psi(x) \tag{61}$$

and the initial condition is $\psi(a) = 1$, $\psi'(a) = 0$. The boundary condition at b retains the general form

$$\alpha\psi(b) + \beta\psi'(b) = 0\,.$$

Equation (61) is seemingly more general than (33) in that the additive term $q(x)\,\psi(x)$ is present. Nevertheless, the analysis here via the discrete polynomial approximations to (61) is essentially unaltered from that of the

case of (33). The conclusion of Theorem 2 carries over and the proofs are the same except for unimportant technical details.

The theory also applies for the initial condition $\varphi(a) = 0$, $\varkappa(a)\,\varphi'(a) = 1$. Here, the discrete approximation is based on the system of orthogonal polynomials of the second kind associated with $\{Q_m^N(\lambda)\}$ (see [5]).

Finally, for certain boundary conditions we may extend the result of Theorem 2 from a finite interval (a, b) to an infinite interval (a, ∞). This is accomplished by executing a limit process from regular eigenvalue problems associated with finite intervals to eigenvalue problems belonging naturally to infinite intervals. The techniques resemble those developed by Coddington and Levinson in Chapter 7 of their classical book on differential equations. We will treat these extensions in another publication.

Remark 2. Under the conditions of Theorem 2 we may execute the same limiting procedure in terms of finite systems of orthogonal polynomials and thus prove that

$$(62) \qquad W\begin{pmatrix} \varphi_0, \varphi_1, \dots, \varphi_{r-1}, \varphi_{i_1}, \varphi_{i_2}, \dots, \varphi_{i_k} \\ x, x, \dots, x, x, x, \dots, x \end{pmatrix} \neq 0 \quad \text{for} \quad a < x < b$$

where $(i_1, i_2, \dots, i_k)$ is a GEB, $i_1 > r-1$, k is even, and r is an arbitrary fixed non-negative integer.

The case r even is already subsumed under Theorem 2. The case of r odd offers a new result. It is important to stress that in the formation of (62) the initial segment of r columns are based on consecutive eigenfunctions culminating with φ_{r-1}; the subsequent columns are based on eigenfunctions with indices comprising a GEB.

The proof of (62) runs parallel to that of (48); we deduce from a limiting argument the fact that (62) does not change sign by appealing to the corresponding result for the polynomial case (see (47)). Then we establish that strict inequality takes place by adapting the proof of Theorem 2.

Remark 3. By the same limiting arguments and under the same conditions as in Theorem 2, we can prove the following inequalities.

(i) Let k be a fixed odd integer, then

$$(63)\quad \operatorname{sign} W\begin{pmatrix}\varphi_n, \varphi_{n+1}, \ldots, \varphi_{n+k-1}\\ x\ , x\ , \ldots, x\end{pmatrix} = \begin{cases}(-1)^{[k(k-1)]/2} & \text{for } x \text{ near } a\\ (-1)^{n+[k(k-1)]/2} & \text{for } x \text{ near } b\end{cases}\quad x \in (a\ , b)\,.$$

If $n=0$

$$(64)\qquad (-1)^{[k(k-1)]/2}\, W\begin{pmatrix}\varphi_0, \varphi_1, \ldots, \varphi_{k-1}\\ x\ , x\ , \ldots, x\end{pmatrix} > 0\,, \qquad \text{for} \quad a<x<b\,.$$

(ii) Let k be an even fixed integer and let r be a fixed arbitrary integer. We consider the augmented Wronskian system

$$U_n(x) = W\begin{pmatrix}\varphi_r, \varphi_n, \varphi_{n+1}, \ldots, \varphi_{n+k}\\ x\ , x\ , x\ , \ldots, x\end{pmatrix} \qquad n = r+1\ , r+2, \ldots\,.$$

Then

$$(65)\quad \operatorname{sign} U_n(x) = \begin{cases}(-1)^{[k(k+1)]/2}\ , & \text{for } x \text{ near } a\\ (-1)^{n-r-1+[k(k+1)]/2}\ , & \text{for } x \text{ near } b\end{cases}\quad x \in (a\ , b)\,.$$

(iii) Let r be an arbitrary fixed integer. Define

$$V_n(x) = W\begin{pmatrix}\varphi_0, \varphi_1, \ldots, \varphi_{r-1}, \varphi_n\\ x\ , x\ , \ldots, x\ , x\end{pmatrix} \qquad n = r\ , r+1, \ldots,$$

then

$$(66)\quad \operatorname{sign} V_n(x) = \begin{cases}(-1)^{[r(r+1)]/2}\ , & \text{for } x \text{ near } a\\ (-1)^{n-r+[r(r+1)]/2}\ , & \text{for } x \text{ near } b\end{cases}\quad x \in (a\ , b)\,.$$

In the case of orthogonal polynomials the proofs of (i), (ii) and (iii) appear in [1], (see Sections 6, 27 and 29).

Remark 4. Although Theorem 2 treated primarily the case of eigenfunctions belonging to a Sturm-Liouville operator of classical type, the same result can be achieved under proper regularity conditions for the generalized Sturm-Liouville equation (5). This includes, besides (33), examples of totally discrete systems and mixtures of discrete and continuous systems. The implications of this generality will be explored elsewhere.

Remark 5. The result of Theorem 2 generalizes to the case of eigenfunctions belonging to the integral equation

(67)
$$\lambda\varphi(x) = \int_a^b K(x, y)\,\varphi(y)\,dy$$

and

(68)
$$K(x, y) = \begin{cases} \chi(x)\,\Theta(y) & a \leq x \leq y \leq b \\ \chi(y)\,\Theta(x) & a \leq y \leq x \leq b\,. \end{cases}$$

Here we require that

(69)
$$\chi(\xi)\,\Theta(\xi) > 0\,, \qquad \frac{d}{d\xi}\left(\frac{\Theta(\xi)}{\psi(\xi)}\right) < 0 \qquad \text{for} \quad a \leq \xi \leq b\,.$$

Under the condition (69), K defines an oscillating kernel as characterized in Section 2 [3, Chapter 2]. It is known that (67) has a countable spectrum $0<\lambda_0<\lambda_1<\ldots$ and corresponding eigenfunctions $\varphi_0, \varphi_1, \varphi_2, \ldots$ enjoying the properties listed in Proposition B. For oscillating kernels we have Theorem 1 which asserts merely a subcase of the result of Theorem 2. It is an open problem if the result of Theorem 2 is valid generally for eigenfunctions of integral operators defined by general oscillating kernels. However, for kernels of the specific form (68), we can prove that

$$W\begin{pmatrix} \varphi_{i_1}, \varphi_{i_2}, \ldots, \varphi_{i_k} \\ x\;, \;x\;, \ldots, \;x \end{pmatrix}$$

never changes sign where $(i_1, i_2, \ldots, i_k)$ is a GEB and $a \leq x \leq b$. We sketch the arguments. Consider a discrete approximation to (67) of the form

(70)
$$\lambda\varphi(x_m^N) = \frac{1}{N+1}\sum_{n=0}^{N} K(x_m^N, y_n^N)\,\varphi(y_n^N) \qquad m = 0, 1, \ldots, N$$

where

$$x_i^N = a + \frac{i(b-a)}{N+1} \qquad i = 0, 1, \ldots, N.$$

The matrix $A = \| K(x_m^N, y_n^N) \|$ is a Green's matrix in the sense of [3, Chapter 2]. The theory of Green's matrices guarantees $N+1$ distinct positive eigenvalues

$$0 < \lambda_0^N < \lambda_1^N < \ldots < \lambda_N^N.$$

Also the inverse matrix of A is a Jacobi matrix [3, Chapter 2] with eigenvalues $1/\lambda_i^N$, $i = 0, 1, \ldots, N$ and the same eigenfunctions. But the eigenfunctions of a Jacobi matrix are in fact finite orthogonal polynomials. From here on the analysis parallels that leading to (47).

§ 4. Sturm Sets of Determinants of Eigenfunctions.

In this section we construct several types of Sturm sets of functions exhibited as determinants whose elements are eigenfunctions as described before. The methods we employ have been extensively applied on a variety of examples in [1] (see Sections 9 and 26).

We will describe briefly the point of view and the technique in our first example below. The verification of the Sturm properties in our other examples follows a similar procedure and consequently the proofs will be omitted.

The crucial tool which underlies the derivation of the desired Sturm properties is the inequality (48) or, more generally, (62). Also, the assertions of Remark 3 will serve to complete certain arguments. Finally, we will employ the following special case of an identity of Sylvestor.

Let A be a determinant of order l and let $1 \leqq m_1 < m_2 \leqq l$, $1 \leqq n_1 < n_2 \leqq l$. We denote by A_{mn} the determinant of order $l-1$, arising from A by striking out the row m and the column n. Similarly, we denote by $A\begin{Bmatrix} m_1, & m_2 \\ n_1, & n_2 \end{Bmatrix}$ the determinant arising from A by striking out the rows m_1, m_2 and the columns n_1, n_2. Then

$$|A|\, A\begin{Bmatrix} m_1, & m_2 \\ n_1, & n_2 \end{Bmatrix} = \begin{vmatrix} A_{m_1, n_1} & A_{m_1, n_2} \\ A_{m_2, n_1} & A_{m_2, n_2} \end{vmatrix}. \tag{71}$$

We now exhibit three classes of Sturm sets whose terms are suitable determinants composed of eigenfunctions arising from a boundary value problem of a regular Sturm-Liouville equation (33) with initial condition (34) and boundary condition of the form (35).

1. Let k be a fixed odd integer. Consider the sequence of functions

$$V_n(x) = W\begin{pmatrix} \varphi_n, & \varphi_{n+1}, & \ldots, & \varphi_{n+k-1} \\ x, & x & , \ldots, & x \end{pmatrix} \qquad n = 0, 1, 2, \ldots \tag{72}$$

for $a < x < b$.

We assert that these constitute a Sturm sequence, i.e., V_n possesses exactly n zeros in (a, b) and zeros of successive functions V_n and V_{n+1} strictly interlace.

For simplicity of exposition we assume that the eigenfunctions $\varphi_i(x)$ are infinitely continuously differentiable and so (72) may be interpreted in the sense of (3). The more general formulation (4) can be handled by appropriate modifications which will be left for the reader.

By virtue of equation (64), Remark 1, we obtain that V_0 is of constant sign for x interior to $[a, b]$.

Invoking the identity (71), we have

$$W\begin{pmatrix} \varphi_{n-1}, \varphi_n, \varphi_{n+1}, \dots, \varphi_{n+k-1} \\ x \quad, x, x \quad, \dots, x \end{pmatrix} \cdot W\begin{pmatrix} \varphi_n, \varphi_{n+1}, \dots, \varphi_{n+k-2} \\ x, x \quad, \dots, x \end{pmatrix} = \begin{vmatrix} V_{n-1}(x) & V_n(x) \\ V'_{n-1}(x) & V'_n(x) \end{vmatrix}. \tag{73}$$

Each Wronskian determinant appearing in the left-hand side is of even order; the sign of the first factor is $(-1)^{[k(k+1)]/2}$ and the sign of the second factor is $(-1)^{[k(k-1)]/2}$. Hence, (73) is strictly negative throughout $a<x<b$. We immediately see on the basis of (73) that if $V_n(x_0)=0$ then

$$V_{n-1}(x_0)\, V'_n(x_0) < 0 \tag{74}$$

and

$$V'_n(x_0)\, V_{n+1}(x_0) > 0\,. \tag{75}$$

It follows trivially that every zero of $V_n(x)$ is a simple zero. Moreover, these relations imply that the zeros of the successive terms $V_n(x)$ and $V_{n+1}(x)$ strictly interlace.

Taking account of (63), we deduce that $V_n(x)$ has precisely n zeros. To sum up

Theorem 3: Let $\varphi_i(x)$, $i=0, 1, \dots$, denote the succesive eigenfunctions belonging to the Sturm-Liouville equation (33) and corresponding initial and boundary conditions (34) and (35) respectively. The set of functions (72) constitute a Sturm sequence, i.e., $V_n(x)$ has exactly n simple zeros and zeros of successive functions $V_n(x)$ and $V_{n+1}(x)$ strictly interlace.

The arguments leading to Theorem 3 apply mutatis mutandis (with one gap) to the case of a Wronskian determinant based on eigenfunctions belonging to integral equations whose kernel is a general oscillating function. The lacuna is that we do not have available the weaker result that $V_n(x)$ has at least n simple zeros. The interlacing character of the zeros belonging to successive functions still holds.

2. By analogous techniques and suitable adaptation of the arguments in [1, Section 27] with reference to (62) and (66) we obtain the following theorem.

Theorem 4: Let the hypothesis of Theorem 2 prevail. Consider the set of functions

$$H_n(x) = W\begin{pmatrix} \varphi_0, \varphi_1, \varphi_2, \dots, \varphi_{r-1}, \varphi_n \\ x, x, x, \dots, x, x \end{pmatrix} \qquad n = r, r+1, \dots \tag{76}$$

where r is a fixed integer which may be even or odd. The sequence (76) form a Sturm set such that $H_n(x)$ has exactly $n-r$ simple zeros and the zeros of consecutive functions strictly interlace.

3. Another set of determinantal functions with remarkable oscillation properties are the augmented systems

$$U_n(x) = W\begin{pmatrix} \varphi_r, \varphi_n, \varphi_{n+1}, \dots, \varphi_{n+k} \\ x, x, x, \dots, x \end{pmatrix} \tag{77}$$

$n = r+1, r+2, \dots,$ and $a < x < b$. Here r is an arbitrary but fixed integer and k is a fixed even integer and $\{\varphi_i\}$ are eigenfunctions satisfying the hypotheses of Theorem 2. These types of augmented systems were studied for the case of orthogonal polynomials in [1]. The identical considerations and methods apply here as well. We now state the final result without proof.

Theorem 5: Let $\{\varphi_i\}$ satisfy the hypothesis of Theorem 2. The functions (77) constitute a weak Sturm set such that $U_n(x)$ has exactly $n-r-1$ nodal zeros and no other zeros. Moreover, the zeros of successive terms of the sequence (77) strictly interlace.

We express our indebtedness for valuable discussions with Professor G. Szegö of Stanford.

BIBLIOGRAPHY

[1] S. Karlin and G. Szegö, "On certain determinants whose elements are orthogonal polynomials", *Journal d'Analyse Mathématique*, Vol. 8 (1961), pp. 1—157.

[2] W. Feller, "On second order differential operators", *Annals of Math.*, Vol. 61 (1955), pp. 90—105.

[3] F. Gantmacher and M. Krein, *Oscillation Matrices and Kernels and Small Vibrations of Mechanical Systems*, 2nd ed. (in Russian), Moscow, 1950.

[4] S. Karlin, "Pólya type distributions II", *Ann. Math. Stat.*, Vol. 28 (1957), pp, 281—308.

[5] J. A. Shohat and J. D. Tamarkin, "The problem of moments", *Mathematical Surveys*, Nr. 1 (1943).

[6] E.A. Coddington and N. Levenson, *The Theory of Differential Equations*, McGraw-Hill Publishing Co., New York (1953).

[7] S. Karlin, *Total positivity and applications to probability and statistics*, Stanford University Press. To appear.

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(Received January 26, 1962)

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